Pathwise mild solutions and optimal regularity estimates for parabolic SPDEs with adapted coefficients

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Overview

1. Introduction
   - Existing results
2. Pathwise mild solutions
   - Evolution families
   - Adaptedness problem
   - New solution formula
3. Main regularity result
   - Introduction
   - Statement
   - Method of proof
4. Conclusion

Introduction

Let $X_0 = L^p(\mathbb{R}^d; \mathbb{C}^N)$ and $X_1 = W^{2m,p}(\mathbb{R}^d; \mathbb{C}^N)$. Consider

$$du(t) + A(t)u(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + \big(B(t)u(t) + g_n(t)\big)\,\mathrm{d}W_n(t), \quad u(0) = u_0$$

- W space-time white noise
- $A(t, \omega)$ $2m$-th order differential operator, with measurable and adapted coefficients
- $f(t, \omega)$ is X_0-valued, $g(t, \omega)$ is $X_{\frac{1}{2}}$-valued
- $B_n(t, \omega, x)$ m-th order differential operator

Goal: To prove existence, uniqueness of an X_1-valued solution u and estimate its norm by the natural norms in terms of f and g.

- Why L^p-theory ? Sobolev embedding, nonlinear eq., numerics.
- Why A only measurable? Coefficients could depend on process.
- A priori estimates can be used for nonlinear equations.

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Existing results

Monotone operators, Lions method (see Prévôt–Röckner 2007)

- A, B adapted, measurable, monotone, coercive
- X_0 Hilbert space.

L^p-theory of Krylov and collaborators since 1996.

- $A(t,\omega)u(x) = \sum_{i,j=1}^{d} a_{ij}(t,\omega,x)D_iD_ju(x)$ + lower order.
- $B_n(t)u(x) = \sum_{j=1}^{d} b_{jn}(t,x)D_ju(x)$ + lower order.

No higher order equations or systems allowed.

Real interpolation spaces Brzezniak 1995, Da Prato–Lunardi 1998.
Stochastic maximal regularity theory of van Neerven-V.-Weis (AOP'12, SIAM J. Math. Anal. '12):

- A continuous in time and Ω-independent
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Littlewood–Paley type estimates

Special cases of the previous slide:

Theorem (van Neerven–V–Weis 2012)

Let $X = L^p(\mathcal{O})$ and let A have ■. For all G adapted one has

$$\Big\| t \mapsto \int_0^t A^{1/2} e^{(t-s)A} G(s)\,\mathrm{d}W(s) \Big\|_{L^p(\Omega\times\mathbb{R}_+\times\mathcal{O})} \leq C\|G\|_{L^p(\Omega\times\mathbb{R}_+\times\mathcal{O};\ell^2)}.$$

The above is a singular stochastic integral.

■ means: A has a bounded H^∞-calculus.

A lot of theory developed and it can be used as a black box.

In practice: every analytic semigroup generator $-A$ on L^p with reasonable coefficients has this property.

$A = -\Delta$ on $\mathbb{R}^d$: Krylov 1996. Cornerstone of his theory.

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Evolution families

Definition

We say that $A(t)$ generates an evolution family $S(t,s)$ if

1. $S(t,t) = I, \;\; S(t,s)S(s,r) = S(t,r)$
2. $(t,s) \mapsto S(t,s)$ is strongly continuous
3. $D_t S(t,s) = A(t)S(t,s)$ and $D_s S(t,s) = S(t,s)A(s)$ on dense set

One-dimensional case: $S(t,s) = \exp(\int_s^t A(r)\,\mathrm{d}r)$.

Example (Gallarati-V. Pot Analysis 2017, Adv. Diff. Eq. 2017)

$A(t)$ $2m$-order system on $\mathbb{R}^d$ with x-independent coefficients. $A(t)$ generates an evolution family $S(t,s)$ on $L^p(\mathbb{R}^d;\mathbb{C}^N))$ for $p \in (1,\infty)$ and

- boundedness on weighted spaces (A_p-weights)
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Proof of boundedness of $S(t,s)$

Fix $s \in \mathbb{R}$. Then the Fourier multiplier $m(t) = \mathcal{F}S(t,s)$ satisfies

$$\partial_t m(t,\xi) + p(\xi,t)m(t,\xi) = 0, \qquad m(s) = I,$$

where $p(\xi,t) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(t)\xi^\alpha\xi^\beta$.
Using the implicit function theorem and ellipticity of $A(t)$ one can check that for all $\gamma \in \mathbb{N}^d$

$$|\xi^\gamma D^\gamma m(t,\xi)| \leq C_\gamma.$$

Thus $S(t,s)$ is bounded on L^p by the Mihlin multiplier theorem.

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Adaptedness problem

- $A : \mathbb{R}_+ \times \Omega \to \mathcal{L}(X_1, X_0)$ adapted
- $A(t,\omega)$ generates evolution family $S(t,s,\omega)$ for $\omega \in \Omega$.

Consider:

$$du(t) + A(t)u(t)\,\mathrm{d}t = g(t)\,\mathrm{d}W(t), \quad u(0) = 0.$$

Mild solution is given by

$$u(t) = \int_0^t S(t,s)g(s)\,\mathrm{d}W(s)??????$$

Adaptedness problem: $S(t,s)$ is only $\mathcal{F}_t$-measurable.
Recall 1-dim case: $S(t,s) = \exp(\int_s^t A(r)\,\mathrm{d}r)$.

This difficulty has limited the semigroup approach to SPDEs

Possible approaches based on Malliavin calculus considered by:
Alòs, León, Viens, Nualart (1999).

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New solution formula: pathwise mild solution

Let $M(t) = \int_0^t g(r)\,\mathrm{d}W(r)$. Then formally:

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The last formula is called the pathwise mild solution (Pronk-V. JDE'14).

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Content of the paper Pronk-V. 2014:

- The pathwise mild solution is a weak solution in PDE sense
- Semigroup approach to stoch. evol. eq. with adapted A

In the paper: A is Hölder continuous in time. Less is needed:

1. $S(t,s)$ exists
2. $\|S(t,s)A(s)\| \leq C(t-s)^{-1}$ (parabolic).

Can be weakened if g takes values in an intermediate space.

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Pathwise mild solutions

Concept is used in several papers already:

- ω-dependent transformations in (t, x), Krüger–Stannat '14

Quasi-linear SPDE:

$$du + A(u)u\,\mathrm{d}t = f(u)\,\mathrm{d}t + g(u)\,\mathrm{d}W.$$

Replace $A(u)u$ by $A(v)u$, fixed point argument with $L(v) := u$.

- Fernando–Sritharan'15, Mohan–Sritharan '17
- Lizzy–Balachandran–Kim '16
- Lu-Neamtu-Schmalfuss '18, Kühn–Neamtu '18

Difficulty:

- How to control the evolution family generated by $A(v)$

Classical theories: Kato, Tanabe, Acquistapace–Terreni usually give bad dependencies.

Special cases Gallarati-V.: only dependence on d, p and ellipticity.

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Optimal regularity for pathwise mild solutions?

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- $A(t,\omega) = -\sum_{i,j=1}^d a_{ij}(t,\omega,x)D_iD_j$ + lower order (complex a_{ij})
- $B_n(t)u(x) = \sum_{j=1}^d b_{jn}(t,x)D_ju(x)$ + lower order (real-valued b_j)
- a_{ij} and b_{nj} unif. cont. in x, only measurable and adapted in (t,ω)
- $a_{ij} - \frac{1}{2}\sum_{n\geq 1} b_{in}b_{jn}$ is uniformly elliptic
- $f \in L^p(\Omega\times\mathbb{R}_+\times\mathbb{R}^d)$, $g \in L^p(\Omega\times\mathbb{R}_+; W^{1,p}(\mathbb{R}^d;\ell^2))$ adapted

In case $f = 0$ and $u_0 = 0$ one needs L^p-estimates for:

$$A(t)u(t) = \mathrm{Term}_1 + \int_0^t \underbrace{A(t)S(t,s)A(s)^{\frac{1}{2}}}_{\sim (t-s)^{-\frac{3}{2}}} A(s)^{\frac{1}{2}} \sum_{n\geq 1}\int_s^t g_n(r)dW_n(r)\,\mathrm{d}s$$

Combination of singular deterministic/stochastic integral.

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Main regularity result for second order operators

Let $\Omega_T = \Omega \times (0, T)$

Theorem (Portal-V. 2018)

Under the above conditions and $p \in [2, \infty)$, for every $T > 0$ there exists a unique $u \in L^p(\Omega \times (0, T); W^{2,p}(\mathbb{R}^d))$ of (SE). Moreover, the following estimate holds:

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Maximal L^p-regularity

Special case of abstract setting. Also allowed: 2m-th order systems.

Improvement of Krylov's results

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Method of proof

Step 1: reduction to x-independent coeff: perturbation argument. After that the main structural conditions are:

- $B(t) = \sum_{j=1}^{J} b_j(t) C_j$, C_j generate commuting C_0-group
- $A(t) + \frac{1}{2}\sum_{j,k=1}^{J} b_j(t) b_k(t) C_j C_k$ generates an evolution family

Step 2: Reduction to $B = 0$

- Ito formula in $L^p \times L^{p'}$ Brźezniak–Neerven–V.–Weis, JDE'08. Based on transformation:

$$\tilde{u}(t) = \prod_{j=1}^{J} \exp(C_j \zeta_j) u(t), \quad \text{with} \quad \zeta_j = \int_0^t b_j(s)\, \mathrm{d}W(s).$$

Step 3: Deterministic term Gallarati–V. Pot. Analysis 2017

Step 4: Initial value: classical.

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Step 5: Estimate of g part by functional calculus argument.
Let $A_0 = -\varepsilon\Delta$ with ε so small that $A(t) - A_0$ still generates an evolution family $S_0(t,s)$. Then $S(t,s) = e^{(t-s)A_0} S_0(t,s)$.

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$$\leq \ldots + \Big\| \int_0^t A_0^{3/2} e^{(t-s)A_0} \widetilde{S_0}(t,s) \int_s^t A(s)^{\frac12} g(r)\,dW(r)\,\mathrm{d}s \Big\|_{L^p(\mathbb{R}^d)}$$

where $\widetilde{S_0}(t,s) = S_0(t,s)A(s)^{\frac12} A_0^{-1/2}$.
The latter can be estimates by proving boundedness of $I(A_0)$ where

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By the H^∞-calculus of A_0 and Kalton–Weis 2001 it remains to check R-boundedness of

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- BDG type estimates in spaces such as $L^p(\ell^2)$
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Pathwise mild solution:

- There is a semigroup approach to equations with random A
- Quasi-linear SPDEs
- The usual parabolic regularity results can be obtained
- Some results also hold in the case $D(A(t,\omega))$ depends on (t,ω) (e.g. 2nd order A, Neumann conditions)

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- Krylov's theory can be obtained for higher order system
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- Maximal estimates with time-dependent A
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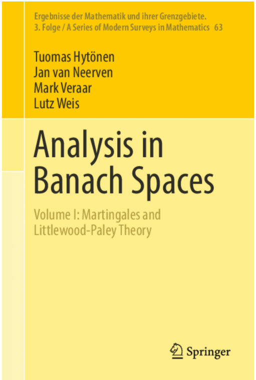

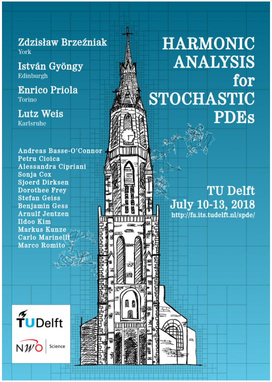

Analysis in Banach spaces
I: Martingales and Littlewood-Paley theory, 2016
II: Probabilistic Techniques and Operator Theory, 2017

Workshop in Delft July 2018:
Harmonic analysis for SPDEs
Information:
`http://fa.its.tudelft.nl/spde/`