Nonlinear diffusion processes and Fokker–Planck–Kolmogorov equations

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Let us consider the following SDE:

$$dx_t = \sqrt{2A(x_t, t)}dw_t + b(x_t, t)\,dt.$$

The measures $\mu_t(B) = P(x_t \in B)$ satisfy the Cauchy problem for the Fokker–Planck–Kolmogorov equation:

$$\partial_t \mu_t = \partial_{x_i}\partial_{x_j}(a^{ij}\mu_t) - \partial_{x_i}(b^i\mu_t), \quad \mu_0 = \nu = \mathrm{Law}(x_0). \qquad (1)$$

A.N.Kolmogorov, W.Feller, ...
Below we consider the case $A = I$:

$$\partial_t \mu_t = \Delta\mu_t - \mathrm{div}(b\mu_t).$$

- If $\langle b(x,t), x\rangle \leq C + C|x|^2$, then there exists a unique probability solution μ_t of the Cauchy problem (1). (Hasminskii R.Z., Bogachev V.I., Da Prato G., M. Röckner, ...)
- If μ_t is a solution of (1) and $b \in L^1([0,T]\times\mathbb{R}^d, \mu_t dt)$, then the corresponding martingale problem has a solution P_ν on $C[0,T]$ and

$$\int \varphi(x)\,\mu_t(dx) = \int_{C[0,T]} \varphi(x(t))dP_\nu.$$

(L.Ambrosio, A.Figalli, D.Trevisan)

- If μ_t and σ_t satisfy (1) with drifts b_μ and b_σ and initial conditions μ_0 and σ_0, then

$$\|\mu_t - \sigma_t\|_{TV} \le \|\mu_0 - \sigma_0\|_{TV} + \Bigl(\int_0^t \int |b_\mu - b_\sigma|^2 \, d\sigma_t \, dt\Bigr)^{1/2}.$$

(Natile L., Peletier M.A., Savare G., Manita O.A., Bogachev V.I., Röckner M, Shaposhnikov S.V.,...)

Bogachev V.I.,Krylov N.V.,Röckner M.,Shaposhnikov S.V. Fokker–Planck–Kolmogorov Equations, Amer. Math. Soc., Providence, Rhode Island, 2015.

Let us consider the Cauchy problem for probability measures μ_t on the $\mathbb{R}^\infty$:

$$\partial_t \mu_t = \sum_{i,j} \partial_{x_i x_j}(a^{ij}\mu) - \sum_i \partial_{x_i}(b^i \mu_t), \quad \mu_0 = \nu,$$

which means that for every $d \geq 1$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int \varphi \, d\mu_t - \int \varphi \, d\nu = \int_0^t \int \Big[\sum_{i,j} a^{ij} \partial_{x_i} \partial_{x_j} \varphi + \sum_i b^i \partial_{x_i} \varphi \Big] \, d\mu_s \, ds.$$

Examples:

$$a^{ii} = \alpha_i^2, a^{ij} = 0 \text{ if } i \neq j, \quad b^i(x) = -\lambda_j^2 x_j + F^i(x).$$

It corresponds to SPDE of the form $du = (\Delta u + F(u))dt + dw_t$, $w_t = \sum_j \alpha_j e_j w_t^j$, $\Delta e_j = -\lambda_j^2 e_j$. (Bogachev V.I., G.Da Prato, M.Rockner, F.Flandoli, ...)

Existence

Set $H_n = \{x\colon x = (x_1, \dots, x_n, 0, 0, \dots)\}$. Let $V, \Theta\colon \mathbb{R}^\infty \to [0, +\infty]$ such that Θ, V are finite on H_n and all the sets $\{\Theta \le R\}$ and $\{V \le R\}$ are compact. Assume that for every $x \in H_n$

$$\sum_{i,j} a^{ij}(x)\partial_{x_i}\partial_{x_j}V(x) + \sum_i b^i(x)\partial_{x_i}V(x) \le V(x) - \Theta(x)V(x)$$

and for every $x \in \mathbb{R}^\infty$

$$|a^{ij}(x)|+|b^i(x)| \le C_i\Big(1+\delta(V(x)\Theta(x))V(x)\Theta(x)\Big), \quad \lim_{s\to 0}\delta(s) = 0.$$

Then for every probability measure ν on $\mathbb{R}^\infty$ with

$$W_1 = \sup_n \int V \circ P_n \, d\nu < \infty, \quad P_n(x) = (x_1, \dots, x_n).$$

the Couchy problem has a solution μ_t such that

$$\int V\, d\mu_t + \int_0^t \int_{\mathbb{R}^\infty} V\Theta \, d\mu_s \, ds \le 4W_1.$$

Uniqueness

Let $a^{ij} = \delta^{ij}$. Let $\mathcal{P}_\nu$ be a convex set of measures $\mu_t(dx)\,dt$ such that $\mu_t(\mathbb{R}^\infty) = 1$, satisfy the Cauchy problem with the initial condition ν and there exists an increasing sequence $N_k \to \infty$ and smooth bounded mappings $\beta_k\colon [0, T] \times \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ such that

$$\int_0^T \int |b(x,t) - b_k(P_{N_k x,t})|^2 \, d\mu_t \, dt \to 0, \quad k \to \infty.$$

The set $\mathcal{P}_\nu$ contains at most one element.
Example: $b^i(x,t) = -\lambda_i^2 x_i + F^i(x)$ and $\|F\|_{l^2} \in L^1(\mu_t\,dt)$.

Bogachev V.I.,Da Prato G.,Röckner M.,Shaposhnikov S.V. An analytic approach to infinite-dimensional continuity and Fokker–Planck–Kolmogorov equations. Annali della Scuola Normale Superiore di Pisa. 2015, V.14, N 3, P. 983–1023.

We consider the Cauchy problem for a nonlinear Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = \Delta \mu_t - \mathrm{div}(b(x, \mu_t)\mu_t), \quad \mu_0 = \nu. \tag{2}$$

and also the corresponding stationary equation

$$\Delta \mu - \mathrm{div}(b(x, \mu)\mu) = 0. \tag{3}$$

It corresponds to the McKean–Vlasov equation

$$dx_t = b(x_t, \mathrm{Law}(x_t))dt + dw_t.$$

M.Kac, H.P. McKean, T. Funaki, ...

The main question: the convergence $\mu_t \to \mu$ if $t \to +\infty$.
Note that it is often simpler, and in the case of a degenerate diffusion matrix more natural, to consider convergence in the Kantorovich metric. Results of this sort for non-gradient drift coefficients were apparently first obtained by N.U. Ahmed and X. Ding, and have been recently generalized by A. Eberle, A. Guillin, R. Zimmer, A. Yu. Veretennikov, F.-Y. Wang. The gradient case, where $b = \nabla V$, has been studied in many papers, starting from D.A. Dawson, J. Gärtner, Y. Tamura and further studied in many papers on the theory of gradient flows by L. Ambrosio, N. Gigli, G. Savaré, F. Bolley, I. Gentil, A. Guillin, J.A. Carrillo, R.J. McCann, C. Villani, ...
Here we discuss convergence in variation.
(O.A. Butkovsky, A. Eberle, ...)

Let us consider the following very typical example demonstrating phenomena arising in the study of convergence of solutions of a nonlinear Fokker–Planck–Kolmogorov equation to the stationary distribution.

Let $d = 1$ and $b(x, \mu) = -x + \varepsilon B(\mu)$, where

$$B(\mu) = \int_{\mathbb{R}} x \, \mu(dx), \quad \varepsilon \geq 0.$$

In case $\varepsilon < 1$ the unique solution of the stationary equation is the standard Gaussian measure μ. One can show that the transition probabilities μ_t forming the solution to the Cauchy problem (2), for every initial condition ν (with a finite first moment), converge exponentially to the stationary measure.

Note that

$$\frac{d}{dt}B(\mu_t) = -(1-\varepsilon)B(\mu_t) \Rightarrow B(\mu_{t+s}) = e^{-(1-\varepsilon)t}B(\mu_s).$$

Let $\tau > 0$. Set

$$\partial_t \sigma_t = \sigma_t'' + (x\sigma_t)', \quad \sigma|_{t=0} = \mu_\tau.$$

We apply the following estimate:

$$\|\mu_{t+\tau} - \sigma_t\|_{TV} \leq \Big(\int_0^t \int |b(x, \mu_{s+\tau}) - b(x, \mu)|^2 \, d\mu_{s+\tau} \, ds \Big)^{1/2},$$

where $t \in [0, \tau]$.

Then

$$\|\mu_{t+\tau} - \sigma_t\|_{TV} \leq \tau C(\varepsilon) B(\mu_\tau).$$

It follows that

$$\|\mu_{2\tau} - \mu\|_{TV} \leq \|\mu_{2\tau} - \sigma_\tau\|_{TV} + \|\sigma_\tau - \mu\|_{TV} \leq C_1 e^{-(1-\varepsilon)\tau/2} + C_2 e^{-C_3\tau}.$$

Thus

$$\|\mu_t - \mu\|_{TV} \leq \alpha_1 e^{-\alpha_2 t}.$$

If $\varepsilon = 1$, then every measure μ given by a density

$$\varrho_a(x) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-|x-a|^2/2\bigr), \quad a \in \mathbb{R}^d,$$

satisfies the stationary equation. It is readily seen that the measures μ_t converge to that stationary measure which has the same mean as ν.

Indeed, in the case under consideration the mean of μ_t does not depend on time and coincides with the mean of ν. Therefore, if the mean of ν coincides with that of μ, then the mean of μ_t coincides with the mean of μ, i.e., $B(\mu_t) = B(\mu)$, and the measures μ_t satisfy the linear Fokker–Planck–Kolmogorov equation corresponding to the Ornstein–Uhlenbeck type operator, for which convergence to the solution of the stationary equation is well known.

Finally, if $\varepsilon > 1$, then a unique stationary solution is the standard Gaussian measure and the solutions to the Cauchy problem converge in the total variation to this measure only if the initial condition ν has zero mean.

Thus, already in this very simple one-dimensional example we see that convergence to the stationary distribution depends not only on the form of nonlinearity, but also on the initial condition, moreover, an important role is played by certain quantities invariant along the trajectories of solutions to the Fokker–Planck–Kolmogorov equation.

Note that the existence of a stationary solution and convergence to it are not stable under small perturbations of the coefficients. For example, if $b(x,\mu) = -x + \delta + \varepsilon B(\mu)$ with arbitrarily small $\delta > 0$, then for $\varepsilon = 1$ there are no stationary solutions, because for the solution μ we must have the equality $(1-\varepsilon)B(\mu) = \delta$.

In the example considered above the coefficient $b(x, \mu)$ has the form

$$b_0(x) + \varepsilon b_1(x, \mu).$$

Convergence of solutions to the Cauchy problem for nonlinear Fokker–Planck–Kolmogorov equations with drift coefficients of such form have been studied in the paper
O.A. Butkovsky, On ergodic properties of nonlinear Markov chains and stochastic McKean–Vlasov equations, Theory Probab. Appl., 58 (2014), 661–674,
where it has been shown that convergence to the stationary distribution takes place in case of a sufficiently small number ε, provided that the coefficients are Lipschitz in x and Lipschitz in μ with respect to the Kantorovich metric. In addition, the term b_1 has been assumed to be globally bounded.

In the paper
A. Eberle, A. Guillin and R. Zimmer, Quantitative Harris type theorems for diffusions and McKean–Vlasov processes, ArXiv:1606.06012 (2016)
a close result has been obtained with the aid of the method of coupling in the case where

$$\langle b_0(x) - b_0(y), x - y\rangle \le -\kappa(|x-y|)|x-y|,$$

$$b_1(x,\mu) = \int_{\mathbb{R}^d} K(x,y)\,d\mu,$$

$$|K(x,y) - K(x',y')| \le C(|x-x'| + |y-y'|),$$

i.e., the global boundedness of b_1 has been weakened by means of the monotonicity condition for b_0.
The smallness of the parameter ε is important not only for convergence, but also for the existence of a stationary distribution, which is seen from the example above.

Let $V \in C^2(\mathbb{R}^d)$, $V \geq 1$ and $\lim_{|x| \to +\infty} V(x) = +\infty$. Let $\mathcal{P}_V(\mathbb{R}^d)$ denote the space of all probability measures μ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} V \, d\mu < \infty.$$

Set

$$W(x) = V(x)^{\gamma}, \quad \gamma \in (0, 1/2].$$

Typical examples are $V(x) = 1 + |x|^{2m}$.

Set

$$\mathcal{M}_\alpha(V) = \Big\{\mu \in \mathcal{P}_V(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} V\,d\mu \le \alpha\Big\}.$$

Let $\|\mu\|_W = \|W\mu\|_{TV}$.
Suppose that for every $\varepsilon \in [0,1]$ we are given a mapping

$$b_\varepsilon(\,\cdot\,,\,\cdot\,)\colon \mathbb{R}^d \times \mathcal{P}_V(\mathbb{R}^d) \to \mathbb{R}^d$$

such that for every $\mu \in \mathcal{P}_V(\mathbb{R}^d)$ the mapping $x \to b_\varepsilon(x,\mu)$ is Borel. Let

$$L_{\mu,\varepsilon}u(x) = \Delta u(x) + \langle b_\varepsilon(x,\mu), \nabla u(x)\rangle.$$

Theorem

Suppose that there exist positive numbers C_1, C_2 and positive functions N_1 and N_2 such that for all $\varepsilon \in [0,1]$, $\alpha > 0$ and $\mu, \sigma \in \mathcal{M}_\alpha(V)$ we have $L_{\mu,\varepsilon}V \le C_1 - C_2 V$ and

$$|b_\varepsilon(x,\mu)| \le N_1(\alpha) V^{1/2-\gamma}(x),$$

$$|b_\varepsilon(x,\mu) - b_\varepsilon(x,\sigma)| \le \varepsilon N_2(\alpha) V(x)^{1/2-\gamma} \|\mu - \sigma\|_W.$$

Then there exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in [0, \varepsilon_0)$ there exists a stationary solution μ and for every measure $\nu \in \mathcal{P}_V(\mathbb{R}^d)$ one has

$$\|\mu_t - \mu\|_W \le \alpha_1 e^{-\alpha_2 t} \quad \forall\, t \ge 0,$$

where $\{\mu_t\}$ is a solution to the Cauchy problem (2) with the initial condition ν and α_1, α_2 are positive numbers such that α_2 does not depend on ν.

Example

Suppose that there exist numbers $m \geq 1$, $\gamma_1 > 0$, $\gamma_2 > 0$ and positive functions N_1, N_2 such that

$$\langle b_\varepsilon(x,\mu), x\rangle \leq \gamma_1 - \gamma_2|x|^2, \quad |b_\varepsilon(x,\mu)| \leq N_1(\alpha)(1+|x|)^m,$$

$$|b_\varepsilon(x,\mu) - b_\varepsilon(x,\sigma)| \leq \varepsilon N_2(\alpha)(1+|x|)^m \|(1+|y|)^m(\mu-\sigma)\|_{TV}$$

for all $\varepsilon \in [0,1]$, $\alpha > 0$ and $\mu, \sigma \in \mathcal{M}_\alpha((1+|x|^{2m+1})$.
Hence all conditions are fulfilled with $V(x) = (1+|x|^2)^{m+1/2})$ and $W(x) = (1+|x|^2)^{m/2}$.

In particular, the listed conditions are fulfilled if

$$b_\varepsilon(x,\mu) = b_0(x) + \varepsilon \int_{\mathbb{R}^d} K(x,y)\,\mu(dy),$$

where

$$\langle b_0(x), x\rangle \le c_1 - c_2|x|^2, \quad \langle K(x,y), x\rangle \le c_3 + c_3|x|^2, \quad c_3 < c_2,$$

$$|b_0(x)| \le c_4 + c_4|x|^m, \quad |K(x,y)| \le c_5(1+|x|^m)(1+|y|^m)$$

with some positive numbers c_1, c_2, c_3, c_4 and c_5.

For a function W as above, let I_0^W denote the set of functions $\psi \in C^2(\mathbb{R}^d)$ such that

$$\sup_x \Big(|\psi(x)| + |\nabla\psi(x)| + |D^2\psi(x)| \Big) W(x)^{-1} < \infty \qquad (4)$$

and for every measure $\mu \in \mathcal{P}_V(\mathbb{R}^d)$ there holds the equality

$$\int_{\mathbb{R}^d} L_\mu \psi \, d\mu = 0. \qquad (5)$$

It is clear that I_0^W is a linear space containing 1.

Further for shortening of writing we use the notation

$$\mu(\psi) := \int_{\mathbb{R}^d} \psi \, d\mu.$$

Proposition

(i) If $\psi \in I_0^W$ and $\{\mu_t\}$ is a solution to the Cauchy problem (2) with the initial condition ν, then $\mu_t(\psi) = \nu(\psi)$.

(ii) If $\mu \in \mathcal{P}_V(\mathbb{R}^d)$ is a solution to the stationary equation (3) and

$$\nu(\psi) \neq \mu(\psi)$$

for some function $\psi \in I_0^W$, then the solutions μ_t to the Cauchy problem (2) with the initial condition ν do not converge to μ with respect to the norm $\| \cdot \|_W$.

Let us consider an important example:

$$b(x,\mu) = -\int_{\mathbb{R}^d} K(x,y)\,\mu(dy),$$

where K is a vector-valued mapping.

Proposition

A function ψ satisfying (4) belongs to I_0^W if and only if for all x, y there holds the equality

$$\Delta\psi(x) + \Delta\psi(y) - \langle K(x,y), \nabla\psi(x)\rangle - \langle K(y,x), \nabla\psi(y)\rangle = 0.$$

In particular, if $\psi \in I_0^W$, then $\Delta\psi(x) - \langle K(x,x), \nabla\psi(x)\rangle = 0$. Moreover, if

$$(Q, K(x,y)) = -(Q, K(y,x))$$

for some constant vector Q and $W(x)$ is growing not slower than $|x|$, then I_0^W contains all functions of the form $\psi(x) = (Q,x) + g$, where g is a constant number.

Let I_+^W denote the set of functions $\psi \in C^2(\mathbb{R}^d)$ such that ψ satisfies condition (4) and there exists a number $\lambda = \lambda(\psi) > 0$ such that

$$\int_{\mathbb{R}^d} L_\mu \psi \, d\mu = \lambda \int_{\mathbb{R}^d} \psi \, d\mu \quad \forall\, \mu \in \mathcal{P}_V(\mathbb{R}^d). \tag{6}$$

Proposition

(i) If $\psi \in I_+^W$ and μ_t is a solution to the Cauchy problem with the initial condition ν, then $\mu_t(\psi) = \nu(\psi) e^{\lambda(\psi) t}$.

(ii) If $\mu \in \mathcal{P}_V(\mathbb{R}^d)$ is a solution to the stationary equation and $\psi \in I_+^W$, then $\mu(\psi) = 0$.

(iii) If $\nu(\psi) \neq 0$ for some $\psi \in I_+^W$, then the solutions μ_t to the Cauchy problem do not converge to the stationary solution with respect to the norm $\| \, \|_W$.

Suppose that for every measure $\nu \in \mathcal{P}_V(\mathbb{R}^d)$ with $\nu|_{I_+^W} = 0$ there exist numbers $C > 0$, $\Lambda > 0$ and $\delta \in [0, 1]$ and a positive function N_1 on $[0, +\infty)$ (thus for different ν these objects can be different) such that

(H_1) for all $\varepsilon \in [0, 1]$, $\alpha \geq 1$ and $\mu \in \mathcal{M}_\alpha(V)$ satisfying the conditions $\mu|_{I_+^W} = 0$ and $\mu|_{I_0^W} = \nu|_{I_0^W}$, there holds the inequality

$$L_{\mu,\varepsilon} V(x) \leq (1 - \delta)C + \Lambda(\delta\alpha - V(x)) \quad \forall x \in \mathbb{R}^d,$$

(H$_2$) for all $\varepsilon \in [0,1]$, α and $\mu \in \mathcal{M}_\alpha(V)$ satisfying the conditions $\mu|_{I_+^W} = 0$ and $\mu|_{I_0^W} = \nu|_{I_0^W}$, there holds the inequality

$$|b_\varepsilon(x,\mu)| \leq N_1(\alpha) V(x)^{\frac{1}{2}-\gamma} \quad \forall x \in \mathbb{R}^d.$$

Suppose that there exists a positive function N_2 on $[0, +\infty)$ such that

(H_3) for all $\varepsilon \in [0, 1]$, all α and all $\mu, \sigma \in \mathcal{M}_\alpha(V)$ satisfying the conditions $\mu|_{I_+^W} = \sigma|_{I_+^W} = 0$ and $\mu|_{I_0^W} = \sigma|_{I_0^W}$, there holds the inequality

$$|b_\varepsilon(x, \mu) - b_\varepsilon(x, \sigma)| \leq \varepsilon N_2(\alpha) V(x)^{\frac{1}{2}-\gamma} \|\mu - \sigma\|_W \quad \forall x \in \mathbb{R}^d.$$

For example, this is the case where $d = 1$ and

$$b_\varepsilon(x,\mu) = -x + \int_{\mathbb{R}} y\,\mu(dy).$$

Here b_ε does not depend on ε. The function x belongs to I_0^W and for every measure μ satisfying the equality

$$\int_{\mathbb{R}} x\,\mu(dx) = \int_{\mathbb{R}} x\,\nu(dx) = Q$$

there hold the estimates $L_{\mu,\varepsilon}(1+|x|^2) \le 3 + Q^2 - (1+|x|^2)$ and

$$|b(x,\mu)| \le |Q| + |x| \le (1+|Q|)(1+|x|^2)^{1/2},$$

i.e., conditions (H_1), (H_2) and (H_3) are fulfilled with

$$C = 2+Q^2, \quad \delta = 0, \quad \Lambda = 1, \quad \gamma = 1/2, \quad N_1 = (1+|Q|), \quad N_2 = 0.$$

Theorem

Suppose that conditions (H_1), (H_2) *and* (H_3) *are fulfilled. Let* $\nu \in \mathcal{P}_V(\mathbb{R}^d)$, $\nu|_{I_+^W} = 0$ *and* $\alpha > 0$. *Then there exist positive numbers* ε_0, α_1 *and* α_2 (*depending on* ν *and* α) *such that, whenever* $\varepsilon \in [0, \varepsilon_0)$, *we have*

$$\|\mu_t - \mu\|_W \le \alpha_1 e^{-\alpha_2 t} \quad \forall\, t \ge 0,$$

where μ_t *is the solution to the Cauchy problem* (2) *with the coefficient* b_ε *and initial data* ν *and* μ *is the stationary solution to equation* (3) *with the coefficient* b_ε *such that*

$$\mu|_{I_0^W} = \nu|_{I_0^W} \quad \text{and} \quad \int_{\mathbb{R}^d} V\, d\mu \le \alpha.$$

Example

Let $d \geq 1$ and

$$b_\varepsilon(x,\mu) = -Rx + \int_{\mathbb{R}^d} \langle v, y\rangle\, \mu(dy) h + \varepsilon \int_{\mathbb{R}^d} H(x,y)\, \mu(dy),$$

where R is a constant matrix, v and h are constant vectors and

$$R^* v = \lambda v, \quad \langle v, h\rangle = \lambda, \quad \langle H(x,y), v\rangle = 0.$$

Suppose also that

$$\langle Rx, x\rangle \geq q|x|^2, \quad q > 0, \quad \sup_{x,y} |H(x,y)| < \infty.$$

For example, for $d = 2$ one can take $R = 2I$, $v = (1, 1)$ and $h = (1, 1)$:

$$b^1_\varepsilon(x, \mu) = -2x_1 + \int_{\mathbb{R}^2} (y_1 + y_2)\,\mu(dy) + \varepsilon \int_{\mathbb{R}^2} H(x, y)\,\mu(dy),$$

$$b^2_\varepsilon(x, \mu) = -2x_2 + \int_{\mathbb{R}^2} (y_1 + y_2)\,\mu(dy) - \varepsilon \int_{\mathbb{R}^2} H(x, y)\,\mu(dy).$$

Then, for every number $Q_0 > 0$, there is a number $\varepsilon_0 > 0$, depending only on Q_0, such that for every $\varepsilon \in [0, \varepsilon_0)$ and $Q \in (-Q_0, Q_0)$ there exists a stationary solution μ for which

$$\int \langle v, y \rangle \, \mu(dy) = Q.$$

Moreover, for every probability measure ν such that $|x|^2 \in L^1(\nu)$ and

$$\int \langle v, y \rangle \, \nu(dy) = Q,$$

the solutions μ_t to the Cauchy problem (2) with initial data ν converge to μ as $t \to +\infty$ and

$$\|(\mu_t - \mu)(1 + |x|)\|_{TV} \leq \alpha_1 e^{-\alpha_2 t}, \quad \alpha_1, \alpha_2 > 0.$$

where α_1 and α_2 depend only on Q_0 and $\|x\|_{L^2(\nu)}$.

The talk is based on the joint paper with V.I.Bogachev and M.Röckner
Convergence in variation of solutions of nonlinear Fokker–Planck–Kolmogorov equations to stationary measures.
2017, Preprint 17027 (CRC 2183) Bielefeld University.

Infinite dimensional nonlinear Fokker–Planck–Kolmogorov equations:

- Existence (Bogachev V.I., Da Prato G., Röckner M., Manita O.A.,...)
- Uniqueness and estimates (Manita O.A.)
- Convergence (????????) The linear case: L.Ambrosio, L.Zambotti,...

THANK YOU!