Conservative stochastic 2-dimensional Cahn-Hilliard equation

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Joint work with Michael Röckner and Huanyu Yang

[Röckner/Yang/Zhu: arXiv 2018]

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Introduction

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$$dX_t = -\frac{1}{2}\Delta\left(\Delta X - :X^3:\right)dt + \nabla \cdot dW_t \quad (1)$$

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$$\nu(d\phi) = c\exp(-\frac{1}{4}\int :\phi^4: dx)\mu(\phi),$$

where c is a renormalized constant and $\mu = \mathcal{N}\left(0, (-\Delta)^{-1}\right)$ is the Gaussian free field. ν is called the ϕ^4_2-quantum field (see [Glimm, Jaffe: 1981])

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Related to Kawasaki dynamics to Ising-Kac model (see [Giacomin, Lebowitz, Presutti99, Mourrat, Weber15])

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  For more regular noise:
- [Da Prato, Debussche:1996]: Obtain a strong solution in 1-3 dimensional case but with space-time white noise or more regular noise.

local well-posedness

Trick from [Da Prato, Debussche03]:
Consider

$$dZ_t = -\frac{1}{2}\Delta^2 Z dt + \nabla \cdot dW_t.$$

$Y = X - Z$, then Y should satisfy the shifted equation:

$$\frac{dY}{dt} = -\frac{1}{2}\Delta^2 Y + \frac{1}{2}\Delta \sum_{k=0}^{3} C_3^k Y^{3-k} : Z^k :$$

and $Z, : Z^k : \in \mathcal{C}^-$.

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Small scales similar as the dynamical Φ_2^4 model
$\Rightarrow$ local well-posedness by fixed point argument for initial value in $\mathcal{C}^{-\frac{4}{3}+}$

Global existence

We split the solution X to (1) into $Y + Z$ where Y satisfies

$$\begin{cases} \frac{dY}{dt} = -\frac{1}{2}\Delta^2 Y + \frac{1}{2}\Delta \sum_{k=0}^{3} C_3^k Y^{3-k} : Z^k : \\ Y(0) = X(0) \end{cases} \qquad (2)$$

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$\Rightarrow$ Global existence with initial value in H^{-1} by compactness argument (not easy to combine local well-posedness argument)

Uniqueness

Suppose u, v are two solutions to (2) and let $r = u - v$, then r satisfies

$$\begin{cases} \dfrac{dr}{dt} = -\dfrac{1}{2}\Delta^2 r + \dfrac{1}{2}\Delta \displaystyle\sum_{k=0}^{2} C_3^k (u^{3-k} - v^{3-k}) : Z^k : \\ r(0) = 0 \end{cases}$$

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$\Rightarrow$ uniqueness with initial value in H^{-1} (using L^4 and H^1-estimates)

Dirichlet form for C-H eq.

The closure of

$$\mathcal{E}^{CH}(u,v) := \frac{1}{2}\int \langle Du, Dv\rangle_{H^{-1}} d\nu,$$

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Remark 1

The Dirichlet form for the dynamical Φ^4_2 model ([Albeverio, Röckner91]) is:

$$\mathcal{E}^{AC}(u,v) := \frac{1}{2}\int \langle \tilde{D}u, \tilde{D}v\rangle_{L^2} d\nu$$

$\tilde{D}u =$ L^2-derivative$= \sum D_{e_k} u e_k$, $\{e_k\}$ is a basis in $L^2(\mathbb{T}^2)$.

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- law is equivalent to the solution obtained by SPDE argument
- $\Rightarrow \Phi_2^4$ field is the invariant measure of the solution given by SPDE argument

Ergodicity

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- Ergodicity by SPDE:
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3-d case

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  No maximal principle/ No $L^p, p > 2$-estimate

Thanks !