Stochastic Partial Differential Equations

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Singular SPDE with rough coefficients

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Setting of this talk

Quasi-linear parabolic PDE driven by f

$\partial_t u - \mathrm{tr}(a(u)D_x^2 u) = \sigma(u)f$

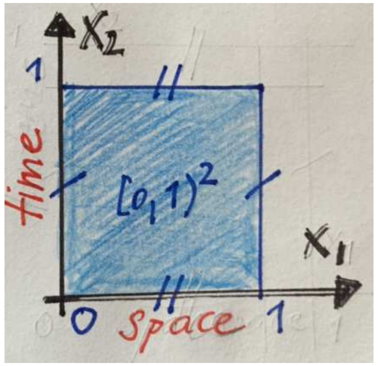
For convenience one space dimension:

$\partial_t u - a(u)\partial_x^2 u = \sigma(u)f$

For simplicity periodic in space and time:

u mean-free and $\partial_t u - P(a(u)\partial_x^2 u + \sigma(u)f) = 0$

Rename coordinates:



$\partial_2 u - P(a(u)\partial_1^2 u + \sigma(u)f) = 0$

x_1 space; x_2 time

The issue

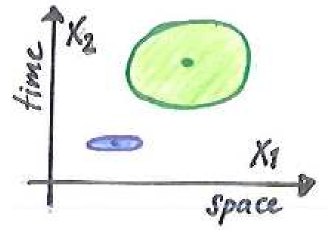
(Small but) very rough "driver" f in

$$\partial_2 u - P(a(u)\partial_1^2 u + \sigma(u) f) = 0$$

parabolic Carnot-Caratheodory metric

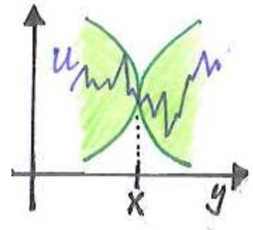
$$d(x,y) := |x_1 - y_1| + \sqrt{|x_2 - y_2|},$$

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(y)-u(x)|}{d^\alpha(y,x)} \quad \text{for } 0 < \alpha < 1.$$



Hölder scale:

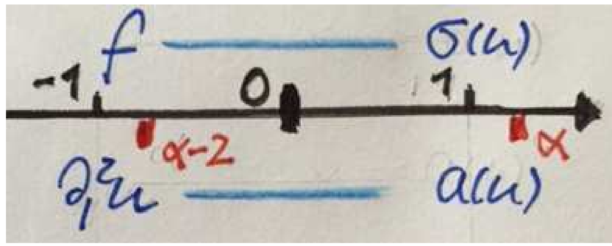
$$C^\alpha := \{[u]_\alpha < \infty\}, \ \ C^{2-\alpha} := \partial_1^2 C^\alpha + \partial_2 C^\alpha.$$



$$f \in C^{\alpha-2} \quad \overset{\text{in best case}}{\Longrightarrow} \quad \partial_1^2 u \in C^{\alpha-2}, \ \sigma(u), a(u) \in C^\alpha.$$

products $a(u)\partial_1^2 u$, $\sigma(u) f$ make sense

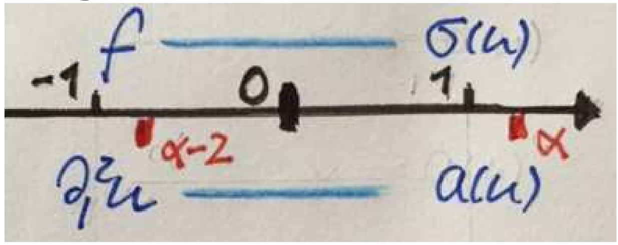
only if $\alpha + (\alpha - 2) > 0 \iff \alpha > 1$.



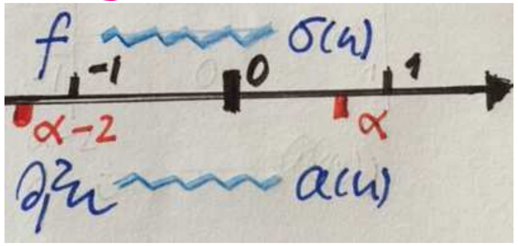
The goal

$\partial_2 u - P(a(u)\partial_1^2 u + \sigma(u) f) = 0$ for $f \in C^{\alpha-2}$

regular case $\alpha > 1$



singular case $\alpha < 1$



Construct (a) solution operator $f \mapsto u$ for $\frac{2}{5} < \alpha < 1$

Hairer & Pardoux '15: for $\frac{2}{5} < \alpha < 1$ but for $a \equiv 1$,
includes f white noise in (x_1, x_2) =KPZ after Hopf-Cole

Extension to quasi-linear: an active area

O.& Weber: *Quasilinear SPDEs via rough paths*
parametric Ansatz, controlled rough path, $\alpha \in (\frac{2}{3}, 1)$

Furlan & Gubinelli: *Para-controlled quasilinear SPDEs*
parametric Ansatz, para-composition, $\alpha \in (\frac{2}{3}, 1)$

Bailleul & Debussche & Hofmanova:
Quasilinear generalized parabolic Anderson model equation
para-products, $\alpha \in (\frac{2}{3}, 1)$

Gerencser & Hairer:
A solution theory for quasilinear singular SPDEs
parametric Ansatz, regularity structures, $(\alpha \in (\frac{1}{2}, \frac{2}{3}))$

O. & Sauer & Smith & Weber:
Parabolic equations with rough coefficients and singular forcing
parametric Ansatz, regularity structures (our twist), $(\alpha \in (\frac{1}{2}, \frac{2}{3}))$,
Today: $\alpha \in (\frac{2}{5}, \frac{1}{2})$, $\sigma \equiv 1$, only deterministic part, drop P

Our flexible approach:
pass via linear equation with rough coefficients

$\partial_2 u - a\diamond\partial_1^2 u = f$ for $f \in C^{\alpha-2}$, $a \in C^{\alpha}$, $\alpha \in (\frac{2}{5}, \frac{1}{2})$

1) semi-concrete solution theory,

solution operator for $\partial_2 - a \diamond \partial_1^2$

within minimal parametric model $v_\alpha, v_{2\alpha}, v_{3\alpha}$

2) abstract integration and reconstruction,

for families of functions $\{U(x,\cdot)\}_x$ and distributions $\{F(x,\cdot)\}_x$

3) concrete solution theory,

targeted towards contraction mapping argument for $a \mapsto u$

(rather $V_a \mapsto V_u$ on level of modelled distributions V)

in case of parametric model $v_0, v_\alpha, w_{2\alpha}, v_1, w_{3\alpha,1}, w_{3\alpha,2}, v_{\alpha+1}$

suitable for nonlinearity $a(u) = u + 1$

Semi-concrete solution theory

Giving sense to $a \diamond \partial_1^2$

Developing a solution theory for $\partial_2 - a \diamond \partial_1^2$

model and off-line assumptions

main result and its two ingredients:
integration & reconstruction

$a \mapsto u$, buckling in perturbative regime

The minimal parametric model in a nutshell

Parameter $a_0 \in [\lambda, \frac{1}{\lambda}]$: placeholder for $a(x)$

Level α: distribution f and functions $\{v_\alpha(\cdot, a_0)\}_{a_0}$

$(\partial_2 - a_0\partial_1^2)v_\alpha = f,$

Level 2α: distrib's $\{a\diamond\partial_1^2 v_\alpha(\cdot, a_0)\}_{a_0}$ and funct's $\{v_{2\alpha}(\cdot, a_0)\}_{a_0}$

$(\partial_2 - a_0\partial_1^2)v_{2\alpha} = a \diamond \partial_1^2 v_\alpha$

Level 3α: distrib's $\{a\diamond\partial_1^2 v_{2\alpha}(\cdot, a_0)\}_{a_0}$ and funct's $\{v_{3\alpha}(\cdot, a_0)\}_{a_0}$

$(\partial_2 - a_0\partial_1^2)v_{3\alpha} = a \diamond \partial_1^2 v_{2\alpha}$

The model: off-line assumptions and integration

Level α. Given distribution f with $\|f\|_{\alpha-2} \le N$.

Let function $v_\alpha(\cdot, a_0)$ satisfy $(\partial_2\text{-}a_0\partial_1^2)v_\alpha = f$.

Standard Schauder: $\|v_\alpha\|_{\alpha,3} \lesssim N$. Notation $\|\cdot\|_{\cdot,3} = C^3$ wrt a_0.

Level 2α. Given fctn. $a \in [\lambda, \frac{1}{\lambda}]$, distr. $a \diamond \partial_1^2 v_\alpha(\cdot, a_0)$ with $\|a\|_\alpha \le N$, $\|[a,(\cdot)] \diamond \partial_1^2 v_\alpha\|_{2\alpha-2,2} \le N^2$.

Notation $\|[a,(\cdot)]\diamond\partial_1^2 v_\alpha\|_{2\alpha-2} := \sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|(a\diamond\partial_1^2 v_\alpha)_T - a(\partial_1^2 v_\alpha)_T\|$.
$(\cdot)_T$ convolution via semigroup $\partial_1^4 - \partial_2^2$ (cf. Bailleul-Bernicot).

Let function $v_{2\alpha}(\cdot, a_0)$ satisfy $(\partial_2\text{-}a_0\partial_1^2)v_{2\alpha} = a \diamond \partial_1^2 v_\alpha$.

Integration: $\|[v_{2\alpha}] - a\partial_{a_0}[v_\alpha]\|_{2\alpha,2} \lesssim N^2$.

Notation $\|[v_{2\alpha}] - a\partial_{a_0}[v_\alpha]\|_{2\alpha} := \sup_{y\ne x} d^{-2\alpha}|[v_{2\alpha}] - a\partial_{a_0}[v_\alpha]|$ with $d := d(y,x)$, $[v] := v(y) - v(x)$. Gubinelli's controlled rough paths

The model: off-line assumptions and integration, cont.

Level 3α. Given distribution $a\diamond\partial_1^2 v_{2\alpha}(\cdot,a_0)$ with $\|[a,(\cdot)]\diamond\partial_1^2 v_{2\alpha}-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2 v_\alpha\|_{3\alpha-2,1}\le N^3$.

Let function $v_{3\alpha}(\cdot,a_0)$ satisfy $(\partial_2\text{-}a_0\partial_1^2)v_{3\alpha}=a\diamond\partial_1^2 v_{2\alpha}$.

Integration: $\|[v_{3\alpha}]-a\partial_{a_0}[v_{2\alpha}]+\frac{1}{2}a^2\partial_{a_0}^2[v_\alpha]\|_{3\alpha,1}\lesssim N^3$.

Notation $\|[v]\|_\beta:=\sup_x\inf_\nu\sup_{y\neq x}d^{-\beta}|[v]-\nu[v_1]|$ for $\beta>1$

Level 4α. Given distribution $a\diamond\partial_1^2 v_{3\alpha}(\cdot,a_0)$ with $\|[a,(\cdot)]\diamond\partial_1^2 v_{3\alpha}-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2 v_{2\alpha}+\frac{1}{2}a^2\partial_{a_0}^2[a,(\cdot)]\diamond\partial_1^2 v_\alpha\|_{4\alpha-2}\le N^4$.

Hierarchy of "norms" $\|\cdot\|_{\alpha-2}$, $\|\cdot\|_{2\alpha-2}$, $\|\cdot\|_{3\alpha-2}$, $\|\cdot\|_{4\alpha-2}$, $\|\cdot\|_{5\alpha-2}$;
$\|\cdot\|_\alpha$, $\|\cdot\|_{2\alpha}$, $\|\cdot\|_{3\alpha}$, $\|\cdot\|_{4\alpha}$

The model: off-line assumptions...

Level α: $\|f\|_{\alpha-2} \leq N$

$(\partial_2\text{-}a_0\partial_1^2)v_\alpha = f \Longrightarrow \|v_\alpha\|_{\alpha,3} \lesssim N.$

Level 2α: $\|[a,(\cdot)] \diamond \partial_1^2 v_\alpha\|_{2\alpha-2,2} \leq N^2.$

$(\partial_2\text{-}a_0\partial_1^2)v_{2\alpha} = a \diamond \partial_1^2 v_\alpha \Longrightarrow \|[v_{2\alpha}] - a\partial_{a_0}[v_\alpha]\|_{2\alpha,2} \lesssim N^2.$

Level 3α: $\|[a,(\cdot)] \diamond \partial_1^2 v_{2\alpha} - a\partial_{a_0}[a,(\cdot)] \diamond \partial_1^2 v_\alpha\|_{3\alpha-2,1} \leq N^3.$

$(\partial_2\text{-}a_0\partial_1^2)v_{3\alpha} = a \diamond \partial_1^2 v_{2\alpha} \Longrightarrow$

$\|[v_{3\alpha}] - a\partial_{a_0}[v_{2\alpha}] + \frac{a^2}{2}\partial_{a_0}^2[v_\alpha]\|_{3\alpha,1} \lesssim N^3.$

Level 4α:

$\|[a,(\cdot)] \diamond \partial_1^2 v_{3\alpha} - a\partial_{a_0}[a,(\cdot)] \diamond \partial_1^2 v_{2\alpha} + \frac{a^2}{2}\partial_{a_0}^2[a,(\cdot)] \diamond \partial_1^2 v_\alpha\|_{4\alpha-2} \leq N^4.$

... outcome of integration

Main semi-concrete result

Solution theory in class

$$u \sim \delta_a.\Big((v_\alpha+v_{2\alpha}+v_{3\alpha})-a\partial_{a_0}(v_\alpha+v_{2\alpha})+\frac{a^2}{2}\partial_{a_0}^2 v_\alpha\Big).$$

Here, $\delta_a=\delta_a(x)$ evaluates a_0 at $a=a(x)$.

Theorem. For $N \ll 1$ $\exists!$ $(u, a\diamond\partial_1^2 u)$ such that

$$\|[u]-\delta_a.\Big([v_\alpha+v_{2\alpha}+v_{3\alpha}]-a\partial_{a_0}[v_\alpha+v_{2\alpha}]+\frac{a^2}{2}\partial_{a_0}^2[v_\alpha]\Big)\|_{4\alpha} \lesssim N^4,$$

$$\|[a,(\cdot)]\diamond\partial_1^2 u-\delta_a.\Big([a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha}+v_{3\alpha})$$
$$-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha})+\frac{a^2}{2}\partial_{a_0}^2[a,(\cdot)]\diamond\partial_1^2 v_\alpha\Big)\|_{5\alpha-2} \lesssim N^5,$$

$$\partial_2 u - a\diamond\partial_1^2 u = f.$$

Main semi-concrete result, spelled out

$\|[u] - \delta_a.\Big([v_\alpha+v_{2\alpha}+v_{3\alpha}]-a\partial_{a_0}[v_\alpha+v_{2\alpha}]+\frac{a^2}{2}\partial_{a_0}^2[v_\alpha]\Big)\|_{4\alpha} \lesssim N^4.$

Setting $\quad U(x,y)$

$:= u(y) - \Big((v_\alpha+v_{2\alpha}+v_{3\alpha})-a_0\partial_{a_0}(v_\alpha+v_{2\alpha})+\frac{a_0^2}{2}\partial_{a_0}^2 v_\alpha\Big)(y,a(x)),$

we have $\quad |U(x,y)-(\nu(x)y_1+c(x))| \lesssim N^4 d^{4\alpha}(y,x)$

$\|[a,(\cdot)]\diamond\partial_1^2 u - \delta_a.\Big([a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha}+v_{3\alpha})$

$-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha})+\frac{a^2}{2}\partial_{a_0}^2[a,(\cdot)]\diamond\partial_1^2 v_\alpha\Big)\|_{5\alpha-2} \lesssim N^5.$

Setting $\quad F(x,\cdot) := -a(x)\partial_1^2 U(x,\cdot) + a\diamond\partial_1^2 u$

$-\Big(a\diamond\partial_1^2(v_\alpha+v_{2\alpha}+v_{3\alpha})-a_0\partial_{a_0}a\diamond\partial_1^2(v_\alpha+v_{2\alpha})+\frac{a_0^2}{2}\partial_{a_0}^2 a\diamond\partial_1^2 v_\alpha\Big)(\cdot,a(x)),$

we have $\quad |F_T(x,x)| \lesssim N^5(T^{\frac14})^{5\alpha-2}.$

Integration step

controlled rough path $O(3\alpha)$ & commutator $O(4\alpha-2)$
$\Longrightarrow$ controlled rough path $O(4\alpha)$

Lemma.

Suppose $\|[u]-\delta_a.\big([v_\alpha+v_{2\alpha}]-a\partial_{a_0}[v_\alpha]\big)\|_{3\alpha}\le {N'}^3$ and
$\|[a,(\cdot)]\diamond\partial_1^2u-\delta_a.\big([a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha})-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2v_\alpha\big)\|_{4\alpha-2}$
$\le N{N'}^3$ for some constant N'.

Then $\|[u]-\delta_a.\big([v_\alpha+v_{2\alpha}+v_{3\alpha}]-a\partial_{a_0}[v_\alpha+v_{2\alpha}]+\frac{a^2}{2}\partial_{a_0}^2[v_\alpha]\big)\|_{4\alpha}$
$\lesssim N{N'}^3+N^4$.

Reconstruction step

controlled rough path $O(4\alpha)$

$\Longrightarrow$ commutator $O(5\alpha-2>0)$

Lemma. Suppose

$$\|[u]-\delta_a.\big([v_\alpha+v_{2\alpha}+v_{3\alpha}]-a\partial_{a_0}[v_\alpha+v_{2\alpha}]+\tfrac{a^2}{2}\partial^2_{a_0}[v_\alpha]\big)\|_{4\alpha} \quad \lesssim N'^4.$$

Then $\exists!$ $a\diamond\partial_1^2 u$ with $\|[a,(\cdot)]\diamond\partial_1^2 u-\delta_a.\big([a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha}+v_{3\alpha})$
$-a\partial_{a_0}[a,(\cdot)]\diamond\partial_1^2(v_\alpha+v_{2\alpha})+\frac{a^2}{2}\partial^2_{a_0}[a,(\cdot)]\diamond\partial_1^2 v_\alpha\big)\|_{5\alpha-2}\lesssim NN'^4+N^5.$

$\|\cdot\|_{3\alpha}\le N'^3$ & $\|\cdot\|_{4\alpha-2}\le NN'^3$

$\Longrightarrow \|\cdot\|_{4\alpha}\lesssim NN'^3+N^4 \qquad \Longrightarrow \|\cdot\|_{3\alpha}\lesssim NN'^3+N^4$

$\Longrightarrow \|\cdot\|_{5\alpha-2}\lesssim N^2N'^3+N^5 \qquad \Longrightarrow \|\cdot\|_{4\alpha-2}\lesssim N^2N'^3+N^5$

Abstract integration and reconstruction

Integration: Safonov's

kernel-free approach to Schauder theory

Reconstruction: Semi-group convolution is handy

Abstract integration: approximation by jets

$(\partial_2 - a(x)\partial_1^2)(u\text{-}U(x,\cdot))$ small $O(\kappa-2)$

& $x \mapsto U(x,\cdot)$ contin. $O(\kappa)$ $\Longrightarrow$ $u - U(x,\cdot)$ small $O(\kappa)$

Lemma. Let $\kappa \in (0,1) \cap (1,2)$ and $\mathsf{A} \subset [0,\kappa)$ finite.

Consider functions u and $\{U(x,\cdot)\}_x$ with

$$|(\partial_2 - a(x)\partial_1^2)(u - U(x,\cdot))_T| \le \sum_{\beta\in\mathsf{A}} d^\beta(\cdot,x)(T^{\frac{1}{4}})^{\kappa-2-\beta}.$$

$$|U(z,\cdot) - U(x,\cdot) - \ell(x,z,\cdot)| \le \sum_{\beta\in\mathsf{A}} d^\beta(z,x) d^{\kappa-\beta}(\cdot,x),$$

where $\ell(x,z,y) = \nu(x,z)y_1 + c(x,z)$.

Then $|u - U(x,\cdot) - \ell(x,\cdot)| \lesssim d^\kappa(\cdot,x)$.

ν only needed for $\kappa > 1$

Safonov's kernel-free jet-based approach to Schauder fits well

Abstract reconstruction: germs of distributions

$x \mapsto F(x,\cdot)$ continuous $O(\kappa)$

$\Longrightarrow \exists\, f$ such that $f - F(x,\cdot)$ small $O(\kappa)$

Lemma. Let $\kappa > 0$ and $\mathsf{A} \subset (0,\kappa)$ finite.

Consider distributions $\{F(x,\cdot)\}_x$ with

$|F_T(x,y) - F_T(y,y)| \leq \sum_\beta d^\beta(y,x)(T^{\frac{1}{4}})^{\kappa-\beta}$.

Then $\exists!$ distribution f with $|f_T(x) - F_T(x,x)| \lesssim (T^{\frac{1}{4}})^{\kappa}$.

Convolution semi-group provides (parabolic) dyadic decomposition

Concrete model

off-line and controlled rough path conditions,
their efficient book-keeping;

notion of modelled distribution;

given modelled distribution for a,
construction of $a\diamond\partial_1^2v_\alpha$, $v_{2\alpha}$, $a\diamond\partial_1^2v_{2\alpha}$, $v_{3\alpha}$,
i.e. concrete $\rightsquigarrow$ semi-concrete

The model in a nutshell

Level 0: $v_0 \equiv 1$; Level 1: $v_1 \equiv x_1$; Level α: v_α as before

Level 2α: $\{v_\alpha \diamond \partial_1^2 v_{\alpha(\cdot, a_0', a_0)}\}$ and $\{w_{2\alpha(\cdot, a_0', a_0)}\}$
with $(\partial_2 - a_0\partial_1^2) w_{2\alpha} = v_\alpha \diamond \partial_1^2 v_\alpha$

Level $3\alpha, 1$: $\{v_\alpha \diamond \partial_1^2 w_{2\alpha(\cdot, a_0'', a_0', a_0)}\}$ and $\{w_{3\alpha,1(\cdot, a_0''=a_0', a_0)}\}$
with $(\partial_2 - a_0\partial_1^2) w_{3\alpha,1} = v_\alpha \diamond \partial_1^2 w_{2\alpha}$

Level $3\alpha, 2$: $\{w_{2\alpha} \diamond \partial_1^2 w_{\alpha(\cdot, a_0'', a_0', a_0)}\}$ and $\{w_{3\alpha,2(\cdot, a_0'', a_0', a_0)}\}$
with $(\partial_2 - a_0\partial_1^2) w_{3\alpha,2} = w_{2\alpha} \diamond \partial_1^2 v_\alpha$

Level $\alpha + 1$: $\{v_{\alpha+1(\cdot, a_0)}\}$ with $(\partial_2 - a_0\partial_1^2) v_{\alpha+1} = v_1 \partial_1^2 v_\alpha$

Ordering of levels: $0 < \alpha < 2\alpha < 1 < 3\alpha < \alpha + 1$

Model: controlled rough path conditions

Controlled rough path conditions on model $v_+ :=$

$$(v_0\,,\ v_{\alpha(a_0)}\,,\ w_{2\alpha(a_0',a_0)}\,,\ v_1\,,\ w_{3\alpha,1(a_0'',a_0',a_0)}\,,\ w_{3\alpha,2(a_0'',a_0',a_0)}\,,\ v_{\alpha+1(a_0)})$$

efficiently encoded via "skeleton" $\Gamma_+ :=$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & -v_\alpha(a_0')\partial_{a_0} & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 0 & 0 & 0 \\ * & \frac{1}{2}v_\alpha(a_0'')v_\alpha(a_0')\partial^2_{a_0} & -v_\alpha(a_0'')\partial_{a_0} & * & 1 & 0 & 0 \\ * & (v_\alpha(a_0'')v_\alpha(a_0')\partial_{a_0'} - w_{2\alpha}(a_0'',a_0'))\partial_{a_0} & -v_\alpha(a_0'')\partial_{a_0'} & * & 0 & 1 & 0 \\ * & -v_1\partial_{a_0} & 0 & * & 0 & 0 & 1 \end{pmatrix}$$

as $\|\Gamma_\beta(x)v_+(y)\|_{\text{level }\beta} \lesssim d^\beta(y,x) \quad \beta \in \{0,\alpha,2\alpha,1,3\alpha,\alpha+1\}$.

Follow by integration from commutator conditions...

Model: commutator conditions

Commutator conditions in terms of product model

$$v := \begin{pmatrix} \partial_1^2 v_\alpha & \partial_1^2 w_{2\alpha} & \partial_1^2 w_{3\alpha,2} & \partial_1^2 v_{\alpha+1} \\ v_\alpha \diamond \partial_1^2 v_\alpha & v_\alpha \diamond \partial_1^2 w_{2\alpha} & v_\alpha \diamond \partial_1^2 w_{3\alpha,2} & v_\alpha \diamond \partial_1^2 v_{\alpha+1} \\ w_{2\alpha} \diamond \partial_1^2 v_\alpha & w_{2\alpha} \diamond \partial_1^2 w_{2\alpha} & & \\ v_1 \partial_1^2 \mathsf{v}_\alpha & v_1 \partial_1^2 \mathsf{w}_{2\alpha} & & \\ w_{3\alpha,1} \diamond \partial_1^2 v_\alpha & & & \\ w_{3\alpha,2} \diamond \partial_1^2 v_\alpha & & & \\ v_{\alpha+1} \diamond \partial_1^2 v_\alpha & & & \end{pmatrix}$$

encoded as $\|\Gamma_\beta(x) v_T(x)\|_{\text{level } \beta} \leq (T^{\frac{1}{4}})^\beta$
for $\beta \in \{\alpha\text{-}2, 2\alpha\text{-}2, 3\alpha\text{-}2, \alpha\text{-}1, 4\alpha\text{-}2, 2\alpha\text{-}1\}$,

where product skeleton $\Gamma_{\beta_+ + \beta_-} := \Gamma_{\beta_+} \otimes \Gamma_{\beta_- + 2}$
with $\beta_+ \in \{0, \alpha, 2\alpha, 1, 3\alpha, \alpha + 1\}$, $\beta_- \in \{\alpha\text{-}2, 2\alpha\text{-}2, 3\alpha\text{-}2, \alpha\text{-}1\}$.

Modelled distribution = augmented description

Modelled distribution (of order 4α) V

$=$ x-dependent *form* on (∞-dim.) space of placeholders

$\mathsf{v}_+ := (\mathsf{v}_0, \mathsf{v}_\alpha(a_0), \mathsf{w}_{2\alpha}(a_0', a_0), \mathsf{v}_1, \mathsf{w}_{3\alpha,1}(a_0'', a_0', a_0), \mathsf{w}_{3\alpha,2}(a_0'', a_0', a_0), \mathsf{v}_{\alpha+1}(a_0)),$

supported on $a_0'' = a_0' = a_0 = a(x)$

Continuity: $|(V(y) - V(x)).\mathsf{v}_+| \le \sum_\beta d^{4\alpha-\beta}(y,x) \|\Gamma_\beta(x)\mathsf{v}_+\|_{\text{level } \beta}$

Gives an augmented description of function $u = V.v_+$
via controlled rough path condition $\|[u] - V.[v_+]\|_{4\alpha} \le 1$.

Next goal: $V_a \mapsto (a \diamond \partial_1^2 v_\alpha\,, \, v_{2\alpha}\,, \, a \diamond \partial_1^2 v_{2\alpha}\,, \, v_{3\alpha});$

combined with previous section: $V_a \mapsto V_u$

for fixed point in case of simplest non-linearity $a(u) = 1 + u$.

Construction of $a \diamond \partial_1^2 v_\alpha$, $v_{2\alpha}$ from V_a

Lemma. $\exists!$ $(v_{2\alpha}, a\diamond\partial_1^2 v_\alpha)$ with

$$\|[v_{2\alpha}] - V_a'.\begin{pmatrix} \partial_{a_0}[v_\alpha] \\ [w_{2\alpha}] \\ [w_{3\alpha,2}] \\ [w_{\alpha+1}] \end{pmatrix}\|_{4\alpha,2} \quad \text{controlled,}$$

$$\|[a,(\cdot)]\diamond\partial_1^2 v_\alpha - V_a.\begin{pmatrix} 0 \\ [v_\alpha,(\cdot)]\diamond\partial_1^2 v_\alpha \\ [w_{2\alpha},(\cdot)]\diamond\partial_1^2 v_\alpha \\ [v_1,(\cdot)]\partial_1^2 \mathrm{v}_\alpha \\ [w_{3\alpha,1},(\cdot)]\diamond\partial_1^2 v_\alpha \\ [w_{3\alpha,2},(\cdot)]\diamond\partial_1^2 v_\alpha \\ [v_{\alpha+1},(\cdot)]\diamond\partial_1^2 v_\alpha \end{pmatrix}\|_{5\alpha-4,2} \quad \text{controlled,}$$

$$(\partial_2 - a_0\partial_1^2)v_{2\alpha} = a\diamond\partial_1^2 v_\alpha.$$

V_a' reduction of V_a to order 3α

Construction of $a \diamond \partial_1^2 v_{2\alpha}$, $v_{3\alpha}$ from V_a

Lemma. $\exists!$ $(v_{3\alpha}, a\diamond\partial_1^2 v_{2\alpha})$ with controlled

$$\|[v_{3\alpha}] - \tfrac{1}{2}(V_a' \otimes V_a')'. \begin{pmatrix} \partial_{a_0}^2[v_\alpha] & \partial_{a_0}[w_{2\alpha}] & \partial_{a_0}[w_{3\alpha,2}] & \partial_{a_0}[v_{\alpha+1}] \\ \partial_{a_0}[w_{2\alpha}] & [2w_{3\alpha,1}] & & \\ \partial_{a_0}[w_{3\alpha,2}] & & & \\ \partial_{a_0}[v_{\alpha+1}] & & & \end{pmatrix} \|_{4\alpha,1},$$

$$\|[a,(\cdot)]\diamond\partial_1^2 v_{2\alpha} - (V_a \otimes V_{\partial_1^2 v_{2\alpha}})'.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ [v_\alpha,(\cdot)]\diamond\partial_1^2 v_\alpha & [v_\alpha,(\cdot)]\diamond\partial_1^2 w_{2\alpha} & [v_\alpha,(\cdot)]\diamond\partial_1^2 w_{3\alpha,2} & [v_\alpha,(\cdot)]\diamond\partial_1^2 v_{\alpha+1} \\ [w_{2\alpha},(\cdot)]\diamond\partial_1^2 v_\alpha & [w_{2\alpha},(\cdot)]\diamond\partial_1^2 w_{2\alpha} & & \\ [v_1,(\cdot)]\partial_1^2 v_\alpha & [v_1,(\cdot)]\partial_1^2 w_{2\alpha} & & \\ [w_{3\alpha,1},(\cdot)]\diamond\partial_1^2 v_\alpha & & & \\ [w_{3\alpha,2},(\cdot)]\diamond\partial_1^2 v_\alpha & & & \\ [v_{\alpha+1},(\cdot)]\diamond\partial_1^2 v_\alpha & & & \end{pmatrix} \|_{5\alpha-4,1},$$

$$(\partial_2 - a_0\partial_1^2)v_{3\alpha} = a \diamond \partial_1^2 v_{2\alpha}.$$

$(V_a \otimes V_{\partial_1^2 v_{2\alpha}})'$ reduction of product to order 0

$(V_a' \otimes V_a')'$ reduction of product to order 4α

Summary and Todo's

$\partial_2 u - a \diamond \partial_1^2 u = f$ for $f \in C^{\alpha-2}$, $a \in C^{\alpha}$, $\alpha \in (\frac{2}{5}, \frac{1}{2})$

1) semi-concrete solution theory,
minimal parametric model v_α , $v_{2\alpha}$, $v_{3\alpha}$

2) abstract integration and reconstruction

3) concrete solution theory,
model v_0 , v_α , $w_{2\alpha}$, $v_1, w_{3\alpha,1}$, $w_{3\alpha,2}$, $v_{\alpha+1}$ targeted to $a(u) = u + 1$

For 3) contraction for $V_a \mapsto V_u$ (ok for self-map property),
general nonlinearity $a(u)$, stability in f (all ok for $\alpha \in (\frac{2}{3}, 1)$).
For 1) & 3) initial (instead of time-periodic) conditions.