Asymptotics for some non-linear stochastic heat equations

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A fractional stochastic heat equation

$$\frac{\partial u_t(x)}{\partial t} = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma(u_t(x))\dot{F}(t,\, x), \quad t > 0,\ x \in \mathbf{R}^d$$

Assumptions :

- $-\nu(-\Delta)^{\alpha/2}$: infinitessimal generator of a symmetric α-stable process with density $p_t(x)$, $\nu > 0$, $\alpha \in (0, 2]$,
- Gaussian noise $\dot{F}(t, x)$: white in time and coloured in space :

$$\mathrm{E}\left(\dot{F}(t,x)\dot{F}(s,y)\right) = \delta_0(t-s)|x-y|^{-\beta}, \qquad 0 < \beta < d$$

- $u_0 : \mathbf{R}^d \to \mathbf{R}_+$ bounded function
- $\sigma : \mathbf{R} \to \mathbf{R}$ globally Lipschitz function (when $\sigma(u) = u$ this is the parabolic Anderson model)

Following Walsh'86, if

$$\beta < \min(\alpha, d),$$

then the equation has a unique mild solution satisfying

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbf{R}^d} p_{t-s}(x-y)\sigma(u_s(y))\, F(\mathrm{d}s\, \mathrm{d}y),$$

where

$$(p_t * u_0)(x) = \int_{\mathbf{R}^d} p_t(x-y)u_0(y)dy,$$

and

$$\sup_{x\in\mathbf{R}^d, t\in[0,T]} \mathrm{E}|u_t(x)|^k < \infty \quad \text{for all} \quad k \geq 2 \quad \text{and} \quad T < \infty.$$

Our first motivation

- Weak comparison principle (Foondun-Joshep-Li'18) : let u and v two solutions such that $u_0 \leq v_0$. Then

$$\mathrm{P}(u_t(x) \leq v_t(x) \quad \text{for all} \quad x \in \mathbf{R}^d,\ t \geq 0) = 1.$$

- This extends (among others) :
  - Mueller'91(space-time white noise, $\alpha = 2$)
  - Xia Chen'16 (case PAM, $\alpha = 2$)
  - Le Chen-Kim'17 (more general u_0, space-time white noise)
  - Le Chen-Huang'18 (measure-valued u_0, $\alpha = 2$)

- If $\sigma(0) = 0$ this result gives non-negativity of the solution.

- The proof uses a system of interacting SDEs with correlated Brownian motion in the lattice and a local limit theorem for stable processes

Our first contribution

- Strong comparison principle (Foondun-Nualart'18) : let u and v two solutions such that $u_0 < v_0$. Then

$$\mathrm{P}(u_t(x) < v_t(x) \quad \text{for all} \quad x \in \mathbf{R}^d,\ t \geq 0) = 1.$$

- This result extends (among others) :
  - Mueller'91(space-time white noise, $\alpha = 2$)
  - Le Chen-Huang'18 (measure-valued initial data, $\alpha = 2$)
  - Le Chen-Kim'17 (more general u_0, space-time white noise)

- We provide a simplification of their proofs and we use the regularization effect of the fractional Laplacian in an essential way.

- Our result could be extended to :
  - more general spatial covariances (as in Le Chen-Huang'18)
  - wider class of initial functions (as in Le Chen-Kim'17 and Le Chen-Huang'18).

Sketch of proof

- It suffices to show that $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbf{R}^d$ a.s.
- Fix $\epsilon > 0$, $R > 0$ and $t > 0$.
- Step 1 : Split the interval $[0, t]$ into subintervals of length $\frac{t}{m}$.

$$\begin{aligned} &\mathrm{P}(u_{s+\epsilon}(x) > 0 \quad \text{for all} \quad 0 \leq s \leq t \quad \text{and} \quad x \in B(0,\ R)) \\ &\geq \lim_{m\to\infty} \mathrm{P}(\cap_{0\leq k\leq m-1} A_k), \end{aligned}$$

where for $k = 0, \dots, m-1$,

$$A_k := \left\{ u_{s+\epsilon}(x) \geq c_1^{k+1} 1_{B_R}(x) \quad \text{for all} \quad s \in \left[\frac{kt}{m}, \frac{(k+1)t}{m}\right] \quad \text{and} \quad x \in \mathbf{R}^d \right\},$$

for some constant $c_1 > 0$.

Sketch of proof

- Two preliminary results :
- Proposition 1 : There exists a constant $0 < c_{\epsilon,R} < 1$ such that for m large enough

$$\int_{\mathbf{R}^d} p_{s+\epsilon}(x-y)u_0(y)\,\mathrm{d}y \geq c_{\epsilon,\,R} \quad \text{for all} \quad x \in B(0,\,R) \quad \text{and} \quad 0 \leq s \leq \frac{t}{m},$$

- Proposition 2 : There exist constants $c_1, c_2 > 0$ such that for m large enough

$$\begin{aligned}\mathrm{P}(u_{s+\epsilon}(x) \geq c_1 1_{B(0,\,R)}(x) \quad \text{for all} \quad 0 \leq s \leq \frac{t}{m} \quad \text{and} \quad x \in \mathbf{R}^d)\\ \geq 1 - \exp\left(-c_2 m^{(\alpha-\beta)/\alpha}[\log m]^{(2\beta-\alpha)/\alpha}\right)\end{aligned}$$

- Step 2 : By Proposition 2 , for m large enough

$$\mathrm{P}(A_0) \geq 1 - \exp(-c_2 m^{(\alpha-\beta)/\alpha}[\log m]^{(2\beta-\alpha)/\alpha})$$

Sketch of proof

- Step 3 : On the event A_{k-1},

$$u_{\frac{kt}{m}+\epsilon}(x) \geq c_1^k 1_{B_R}(x), \quad \text{for all } \; x \in \mathbf{R}^d.$$

- By the Markov property, $\{u_{s+\frac{kt}{m}+\epsilon}(x), s \geq 0, x \in \mathbf{R}^d\}$ solves our equation with the time-shifted noise $\dot{F}_{k,\epsilon}(s, x) := \dot{F}(s + \frac{kt}{m} + \epsilon, x)$ starting from $u_{\frac{kt}{m}+\epsilon}(x)$.

- Let $\{v^{(k)}_{s+\epsilon}(x), s \geq 0, x \in \mathbf{R}^d\}$ be the solution to our equation with the time-shifted noise $\dot{F}_{k,\epsilon}(s, x)$, σ replaced by $\sigma_k(x) = c_1^{-k}\sigma(c_1^k x)$, and initial condition $1_{B_R}(x)$.

- By the Markov property and the weak comparison principle, on A_{k-1},

$$u_{s+\frac{kt}{m}+\epsilon}(x) \geq c_1^k v^{(k)}_{s+\epsilon}(x) \quad \text{for all} \quad x \in \mathbf{R}^d, s \geq 0.$$

- Step 4 : Since

$$\mathrm{P}\big(v_{s+\epsilon}^{(k)}(x) \geq c_1 1_{B_R}(x) \quad \text{for all} \quad s \in \big[0, \frac{t}{m}\big] \quad \text{and} \quad x \in \mathbf{R}^d\big)$$
$$\geq 1 - \exp(-c_2 m^{(\alpha-\beta)/\alpha}[\log m]^{(2\beta-\alpha)/\alpha}),$$

whenever m is large enough, we conclude that

$$\mathrm{P}(A_k | A_{k-1} \cap \cdots \cap A_0) \geq 1 - \exp(-c_2 m^{(\alpha-\beta)/\alpha}[\log m]^{(2\beta-\alpha)/\alpha}).$$

As a consequence,

$$\lim_{m \to \infty} \mathrm{P}(\cap_{0 \leq k \leq m-1} A_k) = 1,$$

which concludes the proof.

A more quantitative result

- Theorem (Foondun-Nualart'18) : Assume $\sigma(0) = 0$. Let $T > 0$ and $K \subset \mathbf{R}^d$ be a compact set. Then, there exist constants $c_1, c_2 > 0$ depending on T and K such that for all $\epsilon > 0$,

$$\mathrm{P}\left(\inf_{t\in[0,\,T]}\inf_{x\in K} u_t(x) < \epsilon\right) \leq c_2 \exp\left(-c_1\{|\log\epsilon|\log|\log\epsilon|\}^{\frac{2\alpha-\beta}{\alpha}}\right).$$

- This result extends Conus-Joshep-Khoshnevisan'12 (space-time white noise, $\alpha = 2$).

- The proof follows similarly using the strong Markov property and the weak comparison principle.

Our second motivation

- Moment comparison theorem (Foondun-Joshep-Li'18) : let u, v two solutions, one with σ, the other with another globally Lipschitz function $\bar{\sigma}$ such that $\bar{\sigma}(0) = \sigma(0) = 0$ and $\sigma(x) \geq \bar{\sigma}(x) \geq 0$ for all $x \in \mathbf{R}_+$. Then for any $k \in \mathbb{N}$, $x \in \mathbf{R}^d$, $t \geq 0$,

$$\mathrm{E}[u_t(x)^k] \geq \mathrm{E}[v_t(x)^k].$$

- This result extends Joshep-Khoshnevisan-Mueller'17 (space-time white noise).

An immediate consequence

From this theorem and the moment estimates in the preprint by Kim'18 for PAM, we have that if

$$\sigma(0) = 0 \quad \text{and} \quad l_\sigma |x| \leq \sigma(x), \qquad \text{for all} \quad x \in \mathbf{R}^d, \tag{1}$$

and u_0 is bounded below, that is,

$$0 < \underline{u}_0 := \inf_{x \in \mathbf{R}^d} u_0(x) \leq \bar{u}_0 := \sup_{x \in \mathbf{R}^d} u_0(x) < \infty, \tag{2}$$

the following sharp moment estimates for the solution hold : there exists $A > 0$ such that for all $x \in \mathbf{R}^d$, $t > 0$, and $k \geq 2$,

$$\frac{\underline{u}_0^k}{A^k} \exp\left(\frac{1}{A} k^{\frac{2\alpha-\beta}{\alpha-\beta}} t \nu^{-\frac{\beta}{\alpha-\beta}} \right) \leq \mathrm{E}|u_t(x)|^k \leq A^k \bar{u}_0^k \exp\left(A k^{\frac{2\alpha-\beta}{\alpha-\beta}} t \nu^{-\frac{\beta}{\alpha-\beta}} \right).$$

Intermittency

- In particular, the solution is fully intermittent meaning that for all $k \geq 2$,

$$k \to \frac{\limsup_{t\to\infty} \log \mathrm{E}|u_t(x)|^k}{k} \quad \text{is strictly increasing.}$$

- Intuitively, this means that the solution develops many high peaks distributed over small x-intervals as t is large

- One aims to better understand the chaotic behaviour of the solution by studying a.s. asymptotic properties depending on the initial data.

Our second contribution

- **Remark** : Since u_0 is bounded

$$\mathrm{E} u_t(x) \leq c,$$

and if u_0 is also bounded below then a.s. $\liminf_{|x|\to\infty} u_t(x)$ is bounded as well.

In contrast, we have :

- **Theorem** (Foondun-Nualart'18) : Under assumptions (1) and (2), there exist constants $c_1, c_2 > 0$ such that for every $t > 0$,

$$c_1 \frac{t^{(\alpha-\beta)/(2\alpha-\beta)}}{\nu^{\beta/(2\alpha-\beta)}} \leq \lim_{R\to\infty} \frac{\log \sup_{x\in B(0,\,R)} u_t(x)}{(\log R)^{\alpha/(2\alpha-\beta)}} \leq c_2 \frac{t^{(\alpha-\beta)/(2\alpha-\beta)}}{\nu^{\beta/(2\alpha-\beta)}} \quad \text{a.s.}$$

- This theorem is a major improvement of Conus-Joshep-Khoshnevisan'13 (PAM, $\alpha = 2$).
- See also Xia Chen'16 for exact spatial asymptotics (fractional-time noise, PAM).

Sketch of proof

- Step 1 : From the sharp moment estimates, we can derive tail estimates :
- Lemma 1 : There exists a constant $c_{A,\alpha,\beta} > 0$ such that for all $\lambda > 0$ and $t > 0$,

$$\sup_{x \in \mathbf{R}^d} \mathrm{P}(u_t(x) > \lambda) \leq \exp\left(-\frac{c_{A,\alpha,\beta}\nu^{\beta/\alpha}}{t^{(\alpha-\beta)/\alpha}} \left| \log \frac{\lambda}{A\bar{u}_0} \right|^{(2\alpha-\beta)/\alpha} \right).$$

- Lemma 2 : There exists a constant $\tilde{c}_{A,\alpha,\beta} > 0$ such that for all $t > 0$ and $k \geq 2$,

$$\inf_{x \in \mathbf{R}^d} (P(u_t(x) > \lambda) \geq \frac{1}{4} \exp\left(-\frac{\tilde{c}_{A,\alpha,\beta}\nu^{\beta/\alpha}}{t^{(\alpha-\beta)/\alpha}} \left(\log \frac{2\lambda A}{\underline{u}_0} \right)^{(2\alpha-\beta)/\alpha} \left(1 + \frac{1}{\log \frac{2\lambda A}{\underline{u}_0}} \right) \right),$$

where $\lambda := \frac{\underline{u}_0}{2A} e^{tk^{\alpha/(\alpha-\beta)}\nu^{-\beta/(\alpha-\beta)}/A}$.

Sketch of proof

Step 2 : We show that if x and x' are $O(1)$ apart, then $u_t(x)$ and $u_t(x')$ are approximately independent.

We construct the approximating sequence as follows. Fix $n \geq 1$. Set $U_t^{(n,0)} := u_0$ and for each $j \geq 1$, the jth Picard iteration is given by

$$U_t^{(n,j)}(x) = \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{B_x((nt)^{\frac{1}{\alpha}})} p_{t-s}(x-y)\sigma(U_s^{(n,j-1)}(y))F^{(n)}(\mathrm{d}s\mathrm{d}y),$$

where $F^{(n)}$ is constructed by multiplying the Riesz kernel by

$$Q_n(x) = \prod_{j=1}^{d} \left(1 - \frac{|x_j|}{n}\right)_+ .$$

- Lemma 3 : Suppose that $\{x_i\}_{i=1}^{\infty} \subset \mathbf{R}^d$ with

$$|x_i - x_j| \geq 2n^{1+1/\alpha} t^{1/\alpha}$$

for all $i \neq j$. Then $\{U^{(n,n)}(x_i)\}_{i=1}^{\infty}$ are independent random variables.

- Lemma 4 : For large enough n,

$$\sup_{t \in [0,T]} \sup_{x \in \mathbf{R}^d} \mathrm{E}|u_t(x) - U_t^{(n,n)}(x)|^k \leq c_2 \frac{1}{n^{\gamma k/2}} e^{c_1 k^{(2\alpha-\beta)/(\alpha-\beta)} t}.$$

Sketch of proof

- Step 3 : Lower bound : Let $t > 0$ and set

$$L := \frac{u_0}{6A} \exp\left[\delta_1 t^{(\alpha-\beta)/(2\alpha-\beta)} |\log R|^{\alpha/(2\alpha-\beta)} \nu^{-\beta/(2\alpha-\beta)}\right],$$

where $\delta_1 > 0$.

- Let $N > 0$ and choose $x_1, x_2, \dots, x_N \in \mathbf{R}^d$ with $|x_i - x_j| \geq 2n^{1+1/\alpha} t^{1/\alpha}$, $i \neq j$.

- One can show that for n, N and R large enough

$$\begin{aligned} \mathrm{P}(\max_{1\leq i\leq N} u_t(x_i) < L) &\leq \left(1 - \mathrm{P}(U_t^{(n,n)}(x_i) \geq 2L)\right)^N + \frac{c}{R^2} \\ &\leq \frac{c}{R^2}, \end{aligned}$$

- Thus

$$\mathrm{P}(\sup_{x\in B(0,\, R)} u_t(x) \leq L) \leq \frac{c}{R^2}.$$

- By Borel Cantelli lemma, a.s. as $R \to \infty$, $\sup_{x\in B(0,\, R)} u_t(x) \geq L$.

Dropping the bounded below assumption

- Example : Let $u_0(x) := 1_{B(0,1)}(x)$. Then for $x \in B(0,\ R)^c$ and R large enough,

$$\mathrm{E} u_t(x) = (p_t * u_0)(x) \leq \frac{ct}{R^\alpha}.$$

  Thus, if we further assume that $\alpha > 1$, a Borel-Cantelli argument shows that

$$\liminf_{|x| \to \infty} u_t(x) = 0.$$

- Thus dropping the bounded below condition can influence the behaviour of the solution drastically.
- The next result looks at the amount of decay u_0 needs to ensure that u_t is a bounded function a.s.

- Assumption : $u_0(x)$ is a radial function satisfying

$$\lim_{x \to \infty} u_0(x) = 0 \quad \text{and} \quad u_0(x) \leq u_0(y) \quad \text{whenever} \quad x \geq y.$$

  Set

$$\Lambda := \lim_{|x| \to \infty} \frac{|\log u_0(x)|}{(\log |x|)^{\alpha/(2\alpha - \beta)}}.$$

Our third contribution

- Theorem (Foondun-Nualart'18) : Under the assumptions above, if $0 < \Lambda < \infty$, there exists a random variable T such that

$$\mathrm{P}\left(\sup_{x \in \mathbf{R}^d} u_t(x) < \infty, \quad \forall t < T \quad \text{and} \quad \sup_{x \in \mathbf{R}^d} u_t(x) = \infty, \quad \forall t > T \right) = 1.$$

If $\Lambda = \infty$, then

$$\mathrm{P}\left(\sup_{x \in \mathbf{R}^d} u_t(x) < \infty, \quad \forall t > 0 \right) = 1.$$

If $\Lambda = 0$, then

$$\mathrm{P}\left(\sup_{x \in \mathbf{R}^d} u_t(x) = \infty, \quad \forall t > 0 \right) = 1.$$

- This result is a major extension of Le Chen-Khoshnevisan-Kim'17 (space-time white noise, $\alpha = 2$). We improve their method of the proof, and ours is based on aGronwall's type result.

Remarks

- Assume $\beta = 1$. Observe that for $\alpha < 2$, $\frac{2}{3} < \frac{\alpha}{2\alpha - 1}$. Choose ϵ such that $\frac{2}{3} < \epsilon < \frac{\alpha}{2\alpha-1}$, and u_0 such that

$$|\log u_0(x)| \sim (\log |x|)^{\epsilon}, \text{ as} |x| \to \infty.$$

  Then $\Lambda = \infty$ if $\alpha = 2$ but $\Lambda = 0$ if $\alpha < 2$.

  Thus, for the same initial data and noise, the solution is bounded for the usual Laplacian but unbounded for the fractional Laplacian.

- When u_0 is constant with compact support, the theorem shows that the solution is bounded for all times a.s.

Key result

- The following insensitivity analysis of the solution with respect to the initial data is a key result of the proof :

- Theorem : Let $a \in \mathbf{R}^d$ and $R > 1$. Let u and v be two solutions with initial conditions u_0 and v_0. Suppose that on $B(a, 2R)$, $u_0(x) = v_0(x)$. Then for all $k \geq 2$ there exist positive constants c_1, c_2 such that for all $t > 0$,

$$\sup_{x \in B(a,\, R)} \mathrm{E}|u_t(x) - v_t(x)|^k \leq c_2 \frac{1}{R^{\alpha k/2}} e^{c_1 k^{(2\alpha-\beta)/(\alpha-\beta)} t}.$$

- This theorem says that when R is large, the values of the solution in $B(a, R)$ are insensitive to the changes of the initial value outside $B(a, R)$.

- For the proof of this result Le Chen-Khoshnevisan-Kim'17 use a technical Lemma from Le Chen-Dalang'15.

Gronwall's type result

- Proposition : Suppose that for $R > 0$, the function $f_R(\cdot)$ is a non-negative non-decreasing locally integrable function on $[0, T]$ satisfying the following

$$f_R(t) \leq A_R(t) + B \int_0^t \frac{f_{2R}(s)}{(t-s)^\gamma}\, ds,$$

where $A_R(\cdot)$ is also a locally integrable non-decreasing function $[0, T]$, B is a positive constant and $\gamma < 1$. If

$$\sup_{n\geq 1} f_{2^n R}(t) < \infty \quad \text{and} \quad A_{(n+1)R}(t) \leq A_{nR}(t) \quad \text{for} \quad n \geq 1,$$

then there exist positive constants c_1, c_2 such that for all $t \in [0, T]$,

$$f_R(t) \leq c_2 A_R(t) e^{c_1 t}.$$

- The proof follows Henry'81.

Tail estimates

- The insensitivity theorem and the tail estimates from the sharp moment estimates give :

- Theorem : Suppose that u and u_0 are as in above. Then, there exist positive constants K_1, K_2 such that for all $\lambda > 0$,

$$-K_1 \frac{\Lambda^{(2\alpha-\beta)/\alpha}}{t^{(\alpha-\beta)/\alpha}} \leq \liminf_{|x|\to\infty} \frac{\log \mathrm{P}(u_t(x) > \lambda)}{\log |x|}$$
$$\leq \limsup_{|x|\to\infty} \frac{\log \mathrm{P}(u_t(x) > \lambda)}{\log |x|} \leq -K_2 \frac{\Lambda^{(2\alpha-\beta)/\alpha}}{t^{(\alpha-\beta)/\alpha}},$$

uniformly for all t in every fixed compact subset of $(0, \infty)$.

- As a consequence, a Borel-Cantelli argument gives the proof of the trichotomy result.

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