Pathwise mild solutions for quasilinear SPDEs

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Joint work with Christian Kühn

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1 Motivation

Lotka-Volterra equations (Volterra, 1929)

$$\begin{cases} \partial_t u_1 = \delta_{11} u_1 - \gamma_{11} u_1^2 - \gamma_{12} u_1 u_2 \\ \partial_t u_2 = \delta_{21} u_2 - \gamma_{21} u_1 u_2 - \gamma_{22} u_2^2. \end{cases}$$

- u_1, u_2 are population densities;
- Long-time behavior: one of the species dies out or $(u_1(t), u_2(t))$ converges for $t \to \infty$ to a steady-state.

1) Lotka-Volterra diffusion system (Mimura, Kishimoto, Weinberger, . . .)

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + \delta_{11} u_1 - \gamma_{11} u_1^2 - \gamma_{12} u_1 u_2 \\ \partial_t u_2 = d_2 \Delta u_2 + \delta_{21} u_2 - \gamma_{21} u_1 u_2 - \gamma_{22} u_2^2. \end{cases}$$

2) Lotka-Volterra cross-diffusion system: SKT model (Shigesada, Kawasaki, Teramoto, 1979)

$$\begin{cases} \partial_t u_1 = \Delta(k_1 u_1 + a u_1 u_2 + c u_2^2) + \delta_{11} u_1 - \gamma_{11} u_1^2 - \gamma_{12} u_1 u_2 \\ \partial_t u_2 = \Delta(k_2 u_2 + b u_1 u_2 + d u_2^2) + \delta_{21} u_2 - \gamma_{21} u_1 u_2 - \gamma_{22} u_2^2 \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 \\ u_1(x,0) = u_0^*(x), \;\; u_2(x,0) = u_0^{**}(x). \end{cases}$$

Quasilinear PDE:

$$du(t) = (\text{div}(P(u)\nabla u) + F(u))dt,$$

where

$$P(u) = \begin{pmatrix} k_1 + a u_2 + 2c u_1 & a u_1 \\ b u_2 & k_2 + 2d u_2 + b u_1 \end{pmatrix}$$

for $u := (u_1, u_2)^T$.

Stochastic SKT model

Let $x \in G$, where $G \subset \mathbb{R}^2$ is an open bounded smooth domain and $t \in [0, T]$.

$$\begin{cases} \partial_t u_1 = \Delta(k_1 u_1 + a u_1 u_2 + c u_2^2) + \delta_{11} u_1 - \gamma_{11} u_2^2 - \gamma_{12} u_1 u_2 + \sigma_1(u_1, u_2) dW_1(t) \\ \partial_t u_2 = \Delta(k_2 u_2 + b u_1 u_2 + d u_2^2) + \delta_{21} u_2 - \gamma_{21} u_1 u_2 - \gamma_{22} u_2^2 + \sigma_2(u_1, u_2) dW_2(t) \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 \\ u_1(x,0) = u_0^*(x), \ \ u_2(x,0) = u_0^{**}(x). \end{cases} \tag{1}$$

Cauchy Problem:

$$\begin{cases} du(t) = (A(u(t))u(t) + F(t, u(t)))dt + \sigma(t, u(t))dW(t), \ \ t \in [0, T] \\ u(0) = u_0. \end{cases}$$

Semigroup approach?

1) Space-time regularity results.
2) Stochastic dynamics: long-time behaviour, attractors.

Quasilinear SPDEs

- W. Liu, M. Röckner, SPDEs in Hilbert space with locally monotone coefficients, 2010.
- M. Hofmanová and T. Zhang. Quasilinear parabolic stochastic partial differential equations: Existence, uniqueness, 2017.
- L. Hornung. Quasilinear parabolic stochastic evolution equations via maximal L^p-regularity, 2017.

Also investigated: weak, martingale, kinetic solutions for quasilinear SPDEs.

I. Semigroup approach: deterministic setting

Let X, Y, Z be three Hilbert spaces such that $Z \hookrightarrow Y \hookrightarrow X$, $0 < R < \infty$ and $K := \{u \in Z : ||u||_Z < R\}$.

Assumptions Cauchy problem:

$$\begin{cases} du(t) = (A(u(t))u(t) + f(t))dt, \quad t \in [0, T] \\ u(0) = u_0 \in Z. \end{cases} \tag{2}$$

A1) $A(u)$ is a sectorial operator for $u \in K$;

A2) $||(\lambda - A(u))^{-1}||_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}$, for $\lambda \in \rho(A(u))$ and $u \in K$;

A3) there exists $\nu \in (0, 1]$ such that

$$||A^{\nu}(u)(A(u)^{-1} - A(v)^{-1})||_{\mathcal{L}(X)} \leq L(R)||u - v||_Y, \text{ for } u, v \in K. \tag{3}$$

- SKT equation: $\nu = 1$, $X = L^2(G)$, $Z := H^{1+\varepsilon}(G)$ and $Y := H^{1+\varepsilon_1}(G)$, $\varepsilon_1 < \varepsilon$.

$$||A(u) - A(v)||_{\mathcal{L}(D(A),X)} \leq L(R)||u - v||_Y.$$

- H. Amann, *Quasilinear evolution equations and parabolic systems*, 1986.
- A. Yagi, *Abstract parabolic evolution equations and their applications*, 2010.

II. Semigroup approach

Let $v \in C([0,T];Z) \cap C^{\mu}([0,T];Y)$, set $A_v(t) := A(v(t))$ and rewrite (2) as

$$\begin{cases} du(t) = (A_v(t)u(t) + f(t))dt, & t \in [0,T] \\ u(0) = u_0 \in Z. \end{cases}$$

1) $U^v(t,t) = \mathrm{Id}$;
2) $U^v(t,r)U^v(r,s) = U^v(t,s)$.

Mild solution:

$$u(t) = U^v(t,0)u_0 + \int_0^t U^v(t,s)f(s)ds.$$

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Stochastic Cauchy problem:

Let $\sigma : [0,T] \to \mathcal{L}_2(H,Z)$ and W stand for an H-cylindrical Brownian motion.

$$\begin{cases} du(t) = (A(u(t))u(t) + f(t))dt + \sigma(t)dW(t), & t \in [0,T] \\ u(0) = u_0 \in Z. \end{cases} \tag{4}$$

Mild solution?

$$u(t) = U^v(t,0,\omega)u_0 + \int_0^t U^v(t,s,\omega)f(s)ds + \int_0^t U^v(t,s,\omega)\sigma(s)dW(s).$$

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Mild solution? Regard $\omega \mapsto U^v(t,s,\omega)$ is not $\mathcal{F}_s$-measurable, only $\mathcal{F}_t$-measurable!

$$u(t) = U^v(t,0,\omega)u_0 + \int_0^t U^v(t,s,\omega)f(s)ds + \int_0^t U^v(t,s,\omega)\sigma(s)dW(s).$$

Pathwise mild solution (M. Pronk and M. Veraar, 2014)

Consider the nonautonomous stochastic evolution equation

$$du(t) = A(t)u(t) + G(t)dW(t).$$

Then its solution

$$u(t) = \int_0^t A(s)u(s)ds + \int_0^t G(s)dW(s)$$

can be written as

$$u(t) = U(t,0)\int_0^t G(s)dW(s) - \int_0^t U(t,s)A(s)\int_s^t G(r)dW(r)ds.$$

Local mild solution

Theorem

There exists a local pathwise mild solution of (1) $u \in L^2(\Omega; C([0, \widetilde{T}); Z))$.

Idea: Rewrite (4) as

$$\begin{cases} du(t) = (A_v(t)u(t) + f(t))dt + \sigma(t)dW(t), \ t \in [0, \widetilde{T}] \\ u(0) = u_0 \in K \text{ a.s.,} \end{cases}$$

which represents a linear parabolic stochastic Cauchy problem, with time-dependent, random drift.

$$u(t) = U^v(t,0)u_0 + U^v(t,0)\int_0^t \sigma(s)dW(s) + \int_0^t U^v(t,s)f(s)ds$$
$$- \int_0^t U^v(t,s)A_v(s)\int_s^t \sigma(r)dW(r)ds, \text{ a.s.,}$$

where $U^v(t,s)$ is the random parabolic evolution operator generated by A_v.

Proof: Sketch

One can show that the mapping $\Phi(v) := u$

is a contraction. Knowing that

$$\frac{\partial}{\partial \tau} U^{v_1}(t,\tau) U^{v_2}(\tau,s) u_0 = U^{v_1}(t,\tau)(A_{v_2}(\tau) - A_{v_1}(\tau)) U^{v_2}(\tau,s) u_0,$$

we have

$$U^{v_2}(t,s)u_0 - U^{v_1}(t,s)u_0 = \int_s^t U^{v_1}(t,\tau)(A_{v_2}(\tau) - A_{v_1}(\tau)) U^{v_2}(\tau,s) u_0 \, d\tau.$$

- main ingredients: estimates of parabolic evolution operators on Sobolev-spaces;
- local solution with Banach's fixed-point theorem for $\widetilde{T}$ small enough;
- nonlinear case: local Lipschitz conditions on the drift and diffusion coefficients.

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we have e.g. $D(A^{\beta}_{v_1}(t)) = H^{2\beta}(G)$

$$A^{\beta}_{v_1}(t)(U^{v_2}(t,s) - U^{v_1}(t,s))u_0 = \int\limits_s^t A^{\beta}_{v_1}(t)U^{v_1}(t,\tau)(A_{v_2}(\tau) - A_{v_1}(\tau))U^{v_2}(\tau,s)u_0 d\tau$$

$$= \int\limits_s^t A^{\beta-\nu+1}_{v_1}(t)U^{v_1}(t,\tau)A^{\nu}_{v_1}(t)(A^{-1}_{v_1}(\tau) - A^{-1}_{v_2}(\tau))A_{v_2}(\tau)U^{v_2}(\tau,s)u_0 d\tau.$$

- main ingredients: estimates of parabolic evolution operators on Sobolev-spaces;
- local solution with Banach's fixed-point theorem for $\widetilde{T}$ small enough;
- nonlinear case: local Lipschitz conditions on the drift and diffusion coefficients.

Global solution? Not always!

- finite-time blow-up;
- positivity of the solution cannot always be preserved;
- degeneracy.

Example (Blow-up)

$$\begin{cases} du = (\Delta u + \frac{1}{2}\Delta v + u^2 + \frac{\lambda_1}{2} v)dt + \sigma(u)dW(t), & \textit{in } G \times [0,T] \\ dv = (\Delta v + (\lambda_1 + k)v)dt, & \textit{in } G \times [0,T], \\ u|_{\partial G} = v|_{\partial G} = 0, & \textit{for } t \in [0,T] \\ u(x,0) = u_0(x), \ v(x,0) = \phi(x), & \textit{in } G, \end{cases} \tag{5}$$

where

$$\begin{cases} \Delta\phi = -\lambda_1\phi, \text{ in } G; \quad \phi = 0, \text{ on } \partial G; \\ \int\limits_G \phi(x)dx = 1. \end{cases} \tag{6}$$

In this case we let $k > \lambda_1$, set $U := (u, v)^T$ and have

$$A(U) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \Delta \text{ and } f(U) = \begin{pmatrix} u^2 + \frac{\lambda_1}{2} v \\ (\lambda_1 + k) v \end{pmatrix}.$$

Lemma

There exists a finite time T^* such that

$$\lim_{t \to T^*-} \mathbb{E} \sup_{x \in G} u(x,t) = \infty. \tag{7}$$

Since $v(x,t) = e^{kt}\phi(x)$ is the solution of the second equation, the first one results in

$$\begin{cases} du = (\Delta u + u^2)dt + \sigma(u)dW(t) \\ u(0) = u_0(x) \geq 0. \end{cases} \tag{8}$$

- G. Lv and J. Duan, *Impacts of noise on a class of partial differential equations*, 2015.

Example (Blow-up and no positivity!)

$$\begin{cases} du = (\Delta u + \frac{1}{2}\Delta v + u(2\lambda_1 - u))dt + \sigma(u)dW(t), & \text{in } G \times [0,T] \\ dv = (\Delta v + (\lambda_1 + k)v)dt, & \text{in } G \times [0,T], \\ u|_{\partial G} = v|_{\partial G} = 0, & \text{for } t \in [0,T] \\ u(x,0) = u_0(x) \geq 0,\ \ v(x,0) = \phi(x), & \text{in } G. \end{cases} \tag{9}$$

Here we have for $U := (u, v)^T$

$$A(U) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \Delta \quad \text{and} \quad f(U) = \begin{pmatrix} u(2\lambda_1 - u) \\ (\lambda_1 + k)v \end{pmatrix}.$$

Lemma

There exists a finite time T^* such that

$$\lim_{t \to T^*-} \mathbb{E} \sup_{x \in G} u(x,t) = -\infty. \tag{10}$$

Conclusion and Outlook

Conclusion:

- Stochastic PDEs with cross diffusion $\rightsquigarrow$ quasilinear stochastic problems;
- Pathwise mild solution (reason: definition of the Itô-integral);
- Existence of local mild solutions;
- Counterexamples for existence of global solutions.

Outlook:

- singular case.
- dynamics and long-time behaviour.

Quasilinear singular SPDEs:

$$\partial u_t = (a(u))\Delta u + f(u)\xi.$$

- I. Bailleul, A. Debussche, and M. Hofmanová. Quasilinear generalized parabolic Anderson model equation, 2016.
- F. Otto and H. Weber. Quasilinear SPDEs via rough paths, 2016.
- M. Gerencsér, M. Hairer. A solution theory for quasilinear singular SPDEs, 2017.

Outlook

Singular SKT Equation: Quasilinear SPDE and coupled KPZ?

$$\partial u_t = a_1(u)\Delta u + \alpha_1(\nabla u)^2 + \beta_1\nabla u\nabla v + f_1(u,v) + g_1(u,v)\xi^1$$
$$\partial v_t = a_2(v)\Delta v + \alpha_2(\nabla v)^2 + \beta_2\nabla u\nabla v + f_2(u,v) + g_2(u,v)\xi^2.$$

Thank you for your attention!