Scale interactions in stochastic fluid models

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Notations

1. $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}, \quad \mathcal{O} \subseteq \mathbb{R}^3$,
2. density : $\varrho = \varrho(\omega, t, x) \in [0, \infty)$,
3. velocity : $\mathbf{u} = \mathbf{u}(\omega, t, x) \in \mathbb{R}^3$,
4. momentum : $(\varrho\mathbf{u}) = (\varrho\mathbf{u})(\omega, t, x) \in \mathbb{R}^3$.

Stochastic compressible Navier–Stokes system (SCNSS)

Mass balance equation (Continuity equation)

$$\mathrm{d}\varrho + \mathrm{div}(\varrho \mathbf{u})\mathrm{d}t = 0$$

Momentum balance equation (Newton's 2nd Law)

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t = \left[\nu \Delta \mathbf{u} - \nabla p(\varrho)\right]\mathrm{d}t + \Phi(\varrho, \varrho \mathbf{u})\mathrm{d}W,$$

$$\boxed{\Phi(\varrho, \varrho \mathbf{u})\mathrm{d}W \approx \varrho\, \mathrm{d}W_1 + \varrho \mathbf{u}\, \mathrm{d}W_2},$$

$$\boxed{p(\varrho) = a\varrho^{\gamma},\ a > 0,\ \gamma > \frac{3}{2}}.$$

Finite energy weak martingale solution

We consider a weak martingale solution

$$[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho, \mathbf{u}, W]$$

with prescribe initial law Λ that is

- weak in the sense of PDEs;
- weak in the sense of probability;
- for any $t \in [0, T]$, the energy inequality

$$\int_0^t \int_{\mathcal{O}} \left[\frac{\varrho|\mathbf{u}|^2}{2} + \frac{a\varrho^\gamma}{\gamma - 1} \right] \mathrm{d}x\mathrm{d}s + \nu \int_0^t \int_{\mathcal{O}} |\nabla \mathbf{u}|^2 \mathrm{d}x\mathrm{d}s$$
$$\leq \int_{\mathcal{O}} \left[\frac{|(\varrho\mathbf{u})|^2}{2\varrho} + \frac{a\varrho^\gamma}{\gamma - 1} \right](0)\mathrm{d}x + \frac{1}{2} \int_0^t \int_{\mathcal{O}} \sum_{k \in \mathbb{N}} \frac{|\mathbf{g}_k(\varrho, \varrho\mathbf{u})|^2}{\varrho} \mathrm{d}x\mathrm{d}s$$
$$+ M(t)$$

holds a.s. for a martingale $M(t)$ and $\mathbf{g}_k = \Phi e_k$.

Existence of finite energy weak martingale solutions

Theorem (Breit, Hofmanová(2016))

Let $\mathcal{O} = \mathbb{T}^3$. Assume that the noise is smooth in its arguments and is of at least linear growth. If the initial energy is bounded, then there exists a FEWMS to the SCNSS.

Theorem (Smith(2017))

Let $\mathcal{O} \subset \mathbb{R}^3$ bounded. Assume that the noise is smooth in its arguments and is of at least linear growth. If the initial energy is bounded, then there exists a FEWMS to the SCNSS under prescribed Dirichlet boundary condition.

Theorem (M. (2017))

Let $\mathcal{O} = \mathbb{R}^3$. Assume that the noise is smooth and compactly supported in space and is of at least linear growth. If the initial energy is bounded, then there exists a FEWMS to the SCNSS under the far field condition $\varrho \to \overline{\varrho}, \quad \mathbf{u} \to 0$, as $|x| \to \infty$, $\overline{\varrho}$ constant.

Rotating fluids

$$
\begin{aligned}
&\mathrm{d}\varrho + \mathrm{div}(\varrho\mathbf{u})\mathrm{d}t = 0,\\
&\mathrm{d}(\varrho\mathbf{u}) + \Big[\mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \boxed{\varrho(\mathbf{e}_3\times\mathbf{u})} + \nabla p(\varrho)\Big]\mathrm{d}t\\
&= \nu\,\Delta\mathbf{u}\,\mathrm{d}t + \boxed{\varrho\nabla G} + \Phi(\varrho,\varrho\mathbf{u})\mathrm{d}W
\end{aligned}
$$

where $\mathbf{e}_3 = (0,0,1)$ and $G = G(x) \in W^{1,\infty}(\mathbb{R}^3)$.

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Remark

- $\varrho(\mathbf{e}_3 \times \mathbf{u}) \perp \mathbf{u}$;
- $\varrho \nabla G \cdot \mathbf{u} = \sqrt{\varrho} \nabla G \cdot \sqrt{\varrho} \mathbf{u}$.

Thus existence of solution follows as in non-rotating fluid.

The Incompressible Navier-Stokes

$$\mathrm{div}(\mathbf{u}) = 0,$$
$$\mathrm{d}(\mathbf{u}) + [\mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi]\mathrm{d}t = \nu \, \Delta \mathbf{u} \, \mathrm{d}t + \Psi(\mathbf{u})\mathrm{d}W.$$

Definition

We say $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{u}, W]$ is a *weak martingale solution* with initial law Λ if

- it is weak in the sense of PDEs;
- it is weak in the sense of probability.

where π is the associated pressure.

Singular limit for rotating fluids

$$\mathrm{d}\varrho + \mathrm{div}(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \left[\mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\mathrm{Ro}}}\varrho(\mathbf{e}_3 \times \mathbf{u}) + \boxed{\frac{1}{\mathrm{Ma}^2}}\nabla p(\varrho)\right]\mathrm{d}t$$

$$= \nu\, \Delta \mathbf{u}\, \mathrm{d}t + \boxed{\frac{1}{\mathrm{Fr}^2}}\varrho \nabla G + \Phi(\varrho, \varrho \mathbf{u})\mathrm{d}W$$

- Low Rossby : $\mathrm{Ro} \to 0$ (from 3D to 2D);
- Low Mach : $\mathrm{Ma} \to 0$ (from compressible to incompressible);
- Low Froude : $\mathrm{Fr} \to 0$ (from inhomogeneous to homogeneous).

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Theorem (M. (2018))

Let $\mathcal{O} = \mathbb{R}^2 \times (0,1)$, $\mathrm{Fr} = \mathrm{Ro} = \varepsilon$ *and* $\mathrm{Ma} = \varepsilon^m$, $m \gg 1$ *then any family of finite energy weak martingale solution* $[\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P}, \varrho_\varepsilon, \mathbf{u}_\varepsilon, W]_{\varepsilon > 0}$ **converges in probability** *to the* $2D$ *incompressible Navier–Stokes system.*

Idea of proof

- By symmetrization, we can recast the problem on a boundaryless domain $\mathbb{R}^2 \times \mathbb{T}$;
- write $\varrho_\varepsilon \mathbf{u}_\varepsilon = \mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \big\langle \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \big\rangle_{x_3} + \mathcal{R}$ where $\mathbb{I} = \mathcal{Q} + \mathcal{P}$ and $\big\langle f \big\rangle_{x_3} := \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} f \, \mathrm{d}x_3$;
- using dispersive estimates, we gain a.s., $\mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to 0$;
- $\big\langle \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \big\rangle_{x_3}$ is regular and converges strongly to $\mathbf{u}$;
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Remark

1. *thus analysis can only be applied to*

$$\Phi(\varrho, \varrho \mathbf{u}) \mathrm{d}W \approx \mathbf{F}_1(\varrho) \mathrm{d}W_1 + \mathbf{F}_2(\varrho \mathbf{u}) \mathrm{d}W_2.$$

   if $\mathbf{F}_2$ is linear;

2. *dispersion/oscillations of acoustic wave is determined by the 'horizontal boundary condition'.*

Idea of proof (cont)

- Set $\mathbf{Y}_\varepsilon := \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)$. Then by taking vertical averages and using spatial regularization, we gain that the limits of

$$\operatorname{div}\big(\mathbf{Y}_\varepsilon \otimes \mathbf{u}_\varepsilon\big) \text{ and } \big\langle \mathbf{Y}_\varepsilon \big\rangle_{x_3} \times \operatorname{curl}\big\langle \mathbf{Y}_\varepsilon \big\rangle_{x_3}$$

coincide a.s. when tested against divergence-free test functions.

Remark

1. $\operatorname{div}\big(\mathbf{Y}_\varepsilon \otimes \mathbf{u}_\varepsilon\big)$ *is irregular because of* $\mathcal{R}$*;*
2. $\big\langle \mathbf{Y}_{\varepsilon,\kappa} \big\rangle_{x_3} \times \operatorname{curl}\big\langle \mathbf{Y}_{\varepsilon,\kappa} \big\rangle_{x_3}$ *is regular and converges strongly since* curl *is linear;*
3. *in summary, the special geometry which allows the taking of vertical averages helps to pass to the limit in the convective term.*

Idea of proof (cont)

weak convergence + pathwise uniqueness = strong convergence.

- On Polish space : Gyöngy–Krylov characterization of convergence in probability,
- on non-metrizable space : Breit–Feireisl–Hofmanová (2018) convergence in probability.

Summary:

1. compactness by Jakubowski-Skorokhod theorem ⇒ recall fast rotations ⇒ 2D problem ⇒ known uniqueness,
2. combine with B-F-M characterization and gain convergence in probability.

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