On the decay rate of the singular values of bivariate functions

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May 14th, 2018

CIRM Conference on Stochastic Partial Differential Equations

Applications: PDEs with random coefficient

Example

Given $0 < \kappa(\cdot, y) \in L^\infty(D)$, a.e., $y \in \Omega$,

$$\begin{cases} -\nabla \cdot (\kappa(\cdot, y)\nabla u(\cdot, y)) = f & \text{in } D \\ u(\cdot, y) = g & \text{on } \partial D \end{cases}$$

- Popularly used numerical methods: Stochastic Galerkin \ Collocation method
- Requirement: finite noise approximation to $\kappa(x, y)$ by

$$\kappa_M(x; Y) \approx \kappa(x, y) \text{ with } Y \text{ being a parameter of dimension } M.$$

- The number of terms M shows the complexity of the corresponding parameterized PDE.

Applications: Multi-dimensional magnetic particle imaging

Example

The functions $\{\kappa_\ell\}_{\ell=1}^L$ can be expressed using the receive coil sensitivities $\{p_\ell\}_{\ell=1}^L$ and the particles' mean magnetic moment vector $\bar{m} : \Omega \times I \to \mathbb{R}^3$ as $\kappa_\ell = \mu_0 p_\ell^t \dot{\bar{m}}$. This relation follows from Faraday's law and the law of reciprocity. Then the inverse problem is to find the concentration $c : \Omega \to \mathbb{R}^+ \cup \{0\}$ from $\{v_\ell\}_{\ell=1}^L$:

$$v_\ell(t) = \int_\Omega c(x) \underbrace{\int_I a_\ell(t-t')\kappa_\ell(x,t')\mathrm{d}t'\mathrm{d}x}_{:=\kappa(x,t)}, \quad \text{with } \kappa_\ell = \mu_0 p_\ell^t \dot{\bar{m}}.$$

• Singular value decay rate of $\kappa(x,t)$ determines the degree of ill-posedness of the inverse problem above.

Karhunen-Loevè expansion

Let $\kappa(x,\omega) \in L^2(D \times \Omega)$, we aim at utilizing as fewer terms as possible to represent $\kappa(x,\omega)$ as a finite sum of terms.

- Covariance kernel:

$$R(x,x') := \int_\Omega \kappa(x,\omega)\kappa(x',\omega)\mathrm{d}\omega$$

- Covariance operator:

$$\mathcal{R} : L^2(D) \to L^2(D), \quad \mathcal{R}v = \int_D R(x,x')v(x')\mathrm{d}x'.$$

- Eigenpairs of $\mathcal{R}$: seeking $\{\lambda_n, \phi_n\} \in \mathbb{R} \times L^2(D)$, s.t.,

$$\mathcal{R}\phi_n = \lambda_n \phi_n.$$

Reorder the eigenvalues by $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq$

Note that the operator $\mathcal{R}$ is a Hilbert-Schmidt operator, therefore, $\lambda_n \to 0$ as $n \to \infty$.

Karhunen-Loevè expansion

- For all $\lambda_n \neq 0$, define

$$\psi_n(\omega) := \lambda_n^{-1/2} \int_D \kappa(x,\omega)\phi_n(x)\mathrm{d}x.$$

- Karhunen-Loevè expansion:

$$\kappa(x,\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\phi_n(x)\psi_n(\omega)$$

- M-term truncation:

$$\kappa_M(x,\omega) = \sum_{n=1}^{M} \sqrt{\lambda_n}\phi_n(x)\psi_n(\omega).$$

- Truncation estimate: $\kappa_M(x,\omega)$ is optimal in mean square error merely relying on the decay rate of $\{\lambda_n\}_{n=1}^{\infty}$

$$\|\kappa(x,\omega) - \kappa_M(x,\omega)\|_{L^2(\Omega\times D)} = \Big(\sum_{n>M} \lambda_n\Big)^{1/2}.$$

State of art

Table: State of the art of eigenvalue decay results and their corresponding truncation error estimates.

reference	$\kappa(x,\omega)$	λ_n	M-term truncation error	
			Condition	Rate
a	$L^2(\Omega, H^s(D))$	$\mathcal{O}(n^{-\frac{s}{d}})$	$s > d$	$\mathcal{O}(M^{\frac{1}{2}-\frac{s}{2d}})$
b	$L^2(\Omega, H^s(D))$	$\mathcal{O}(n^{-\frac{2s}{d}-\frac{1}{2}})$	$s > \frac{d}{4}$	$\mathcal{O}(M^{\frac{1}{4}-\frac{s}{d}})$
c	$H^s(\Omega \times D)$	$\mathcal{O}(n^{-\frac{2s}{d^*}})$	$s > \frac{d^*}{2}$	$\mathcal{O}(M^{\frac{1}{2}-\frac{s}{d^*}})$
d	$L^2(\Omega, H^s(I))$	-	$s > 0$	$\mathcal{O}(M^{-s})$
results of this article	$L^2(\Omega, \dot{H}^s(D))$ $L^2(\Omega, H^s(D))$	$\mathcal{O}(n^{-1-\frac{2s}{d}})$	$s > 0$	$\mathcal{O}(M^{-\frac{s}{d}})$

a Agmon, Princeton, N.J.-Toronto-London, 1965.
b Pietsch, Cambridge University Press, Cambridge, 1987.
c Griebel and Harbrecht, IMA J. Numer. Anal., 34(1):28-54, 2014.
d Azaïez and Belgacem. Comput. Methods Appl. Mech. Engrg., 290:57-72, 2015.

Motivation

If $\kappa(x,\omega) \in L^2(\Omega, L^2(D))$, then $\{\lambda_n\}_{n=1}^{\infty} \in \ell_1$.

In view that

$$\kappa(x,\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\phi_n(x)\psi_n(\omega),$$

we can obtain

$$\int_{\Omega} \Big(\int_{D} |\kappa(x,\omega)|^2 \mathrm{d}x \Big) \mathrm{d}\omega = \sum_{n=1}^{\infty} \lambda_n < \infty$$

Therefore, the M-term KL truncation κ_M converges to κ in $L^2(\Omega, L^2(D))$.

However, none of the previous results on the decay rate of the eigenvalues guarantee this simple fact.

Main result

Theorem (Eigenvalue estimate for the general space $L^2(\Omega, H^s(D))$)

Assume that D satisfies the strong local Lipschitz condition. Let $\kappa(x,\omega) \in L^2(\Omega, H^s(D))$. Then $\{\lambda_n\}_{n=1}^{\infty} \in \ell_{\frac{d}{d+2s},1}$. In specific, there holds

$$\lambda_n \leq \mathrm{diam}(D)^{2s} C_{\mathrm{em}}(d,s) C_{\mathrm{ext}}(D,s) \|\kappa\|^2_{L^2(\Omega, H^s(D))} n^{-1-\frac{2s}{d}},$$

where $C_{\mathrm{ext}}(D,s)$ is a constant depending only on D and s.

Proof.

Step 1. Prove this result when $\kappa(x,\omega) \in L^2(\Omega, \dot{H}^s(D))$, where

$$\dot{H}^s(D) = \left\{ v \in L^2(D) : \sum_{j=1}^{\infty} \nu_j^s \cdot (v, \xi_j)^2 < \infty \right\}.$$

Here, $\{\nu_j, \xi_j\}_{j=1}^{\infty}$ denotes the eigenpairs of $A := -\Delta|_{H_0^1(D)}$ with nondecreasing eigenvalues.

Step 2. Prove this result in $L^2(\Omega, H^s(I^d))$, where $I := [0,1]$.

Step 3. Then utilizing the extension Theorem to prove the result.

Proof to Step 1(1/3).

The Hilbert space $\dot{H}^s(D)$ is endowed with the inner product

$$(v,w)_s = \sum_{j=1}^{\infty} \nu_j^s (v,\xi_j)(w,\xi_j), \text{ for } v,w \in \dot{H}^s(D).$$

For any $s>0$, one can now define the fractional power operator $T = A^{s/2}$ on $\dot{H}^s(D)$ by

$$Tv = \sum_{j=1}^{\infty} \nu_j^{\frac{s}{2}} \cdot (v,\xi_j) \cdot \xi_j.$$

Consequently, there holds

$$\|Tv\|_{L^2(D)}^2 = \sum_{j=1}^{\infty} \nu_j^s (v,\xi_j)^2 = |v|_s^2 \quad \text{for all } v \in \dot{H}^s(D).$$

Let $\{\mu_j\}_{j=1}^{\infty}$ be the eigenvalues of T in nondecreasing order. Then we can obtain

$$\mu_j \geq C_{\text{weyl}}(d)^{\frac{s}{2}} \operatorname{diam}(D)^{-s} j^{s/d}.$$

Proof to Step 1(2/3).

Let

$$R_T(x,x') := \int_\Omega T\kappa(x',\omega)T\kappa(x,\omega)\mathrm{d}\omega \in L^2(D\times D),$$

and denote by $\mathcal{R}_T : L^2(D) \to L^2(D)$ the Hilbert-Schmidt operator associated with the kernel function $R_T(x,x')$. Let

$$\mathcal{R}_1 = T\mathcal{R}T,$$

with its domain $\mathcal{D}(\mathcal{R}_1) = \dot{H}^s(D)$.

The following statements are valid:

(i) $\mathcal{R}_1 \in \mathcal{B}(\dot{H}^s(D), L^2(D))$.

(ii) $\mathcal{R}_T\,|_{\dot{H}^s(D)} = \mathcal{R}_1$. Hence, $\mathcal{R}_1$ is a symmetric operator with $\mathcal{R}_T$ as its self-adjoint extension operator.

(iii) There holds the identity

$$\|T\kappa\|^2_{L^2(\Omega\times D)} = \sum_{n=1}^{\infty} \lambda_n \|T\phi_n\|^2_{L^2(D)}.$$

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Proof to Step 1(3/3).

For any $L^2(D)$-orthonormal system $\{e_n\}_{n=1}^m \subset \dot{H}^s(D)$ of m elements, there holds

$$\sum_{n=1}^{m} \| Te_n\|_{L^2(D)}^2 \geq \sum_{n=1}^{m} \mu_n^2.$$

Together with the previous result, we arrive at

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n \| T\phi_n\|_{L^2(D)}^2 &= (\lambda_1 - \lambda_2) \| T\phi_1\|_{L^2(D)}^2 + (\lambda_2 - \lambda_3) \left(\| T\phi_1\|_{L^2(D)}^2 + \| T\phi_2\|_{L^2(D)}^2 \right) \\
&\quad + (\lambda_3 - \lambda_4) \left(\| T\phi_1\|_{L^2(D)}^2 + \| T\phi_2\|_{L^2(D)}^2 + \| T\phi_3\|_{L^2(D)}^2 \right) + \cdots \\
&\geq (\lambda_1 - \lambda_2)\mu_1^2 + (\lambda_2 - \lambda_3)(\mu_1^2 + \mu_2^2) + (\lambda_3 - \lambda_4)(\mu_1^2 + \mu_2^2 + \mu_3^2) + \cdots \\
&= \sum_{n=1}^{\infty} \lambda_n \mu_n^2 \geq \sum_{n=1}^{\infty} \lambda_n C_{\mathrm{weyl}}(d)^s \mathrm{diam}(D)^{-2s} n^{\frac{2s}{d}}.
\end{aligned}$$

$$\{\lambda_n\}_{n=1}^{\infty} \in \ell_{\frac{d}{d+2s},1} \quad \text{and} \quad \|\{\lambda_n\}_{n=1}^{\infty}\|_{\ell_{\frac{d}{d+2s},1}} \leq C_{\mathrm{weyl}}(d)^{-s} \mathrm{diam}(D)^{2s} \|\kappa\|_{L^2(\Omega,\dot{H}^s(D))}^2.$$

Numerical simulations[1]

Examples

1. $\kappa(x,y) = \exp(-|x-y|) \in H^{3/2-\epsilon}([0,1]^2)$ for $(x,y) \in [0,1]^2$;
2. $\kappa(x,y) = (1+|x-y|)\exp(-|x-y|) \in H^{7/2-\epsilon}([0,1]^2)$ for $(x,y) \in [0,1]^2$.

Then by our result, $\sqrt{\lambda_n} \approx n^{-2}$ and n^{-4}, respectively.

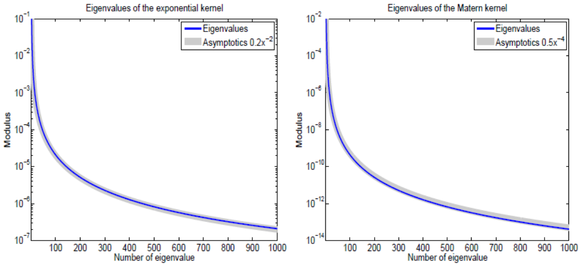

FIG. 5. The eigenvalues of the exponential kernel (left) and the Matérn kernel (right).

[1] Griebel and Harbrecht, *IMA J. Numer. Analysis*, 2012.

Related publications

- Griebel and **GL**, *SIAM J. Numer. Anal., 56(2), 974-993*, 2018.

Acknowledgement to

THE ROYAL SOCIETY
BRITISH ACADEMY
The Academy of Medical Sciences

Thanks for your attention!