Global solutions to elliptic and parabolic Φ^4 models in Euclidean space

Martina Hofmanová

Bielefeld University

joint work with Massimiliano Gubinelli

Elliptic Φ^4 model

$$(-\Delta + \mu)\varphi + \varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}^d \quad d = 4, 5$$

- ξ a space white noise
- $\mu > 0$

Parabolic Φ^4 model

$$(\partial_t - \Delta + \mu)\varphi + \varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^d \quad d = 2, 3$$

- ξ a space-time white noise
- $\mu \in \mathbb{R}$

Connection to Φ^4 Euclidean quantum field theory

- the measure given formally by

$$\nu(d\varphi) \sim \exp\left[-\int \frac{1}{2}|\nabla\varphi|^2 + \frac{\mu}{2}\varphi^2 + \frac{1}{4}\varphi^4 dx\right] d\varphi$$

- **parabolic on** $\mathbb{R}^d$ – linked to Φ^4_d via Parisi–Wu '81 stochastic quantization
  - ν is the invariant measure of the parabolic equation

- **elliptic on** $\mathbb{R}^d$ – linked to Φ^4_{d-2} via Parisi–Sourlas '79 dimensional reduction
  - show that a solution evaluated on a $(d-2)$-dimensional hyperplane has the law ν

$$(-\Delta+\mu)\varphi+\varphi^3=\xi \qquad d=4,5 \qquad\qquad (\partial_t-\Delta+\mu)\varphi+\varphi^3=\xi \qquad d=2,3$$

- ξ space white noise on $\mathbb{R}^d$ - a random distribution locally in $B^{-d/2-\kappa}_{\infty,\infty}$
- ξ space-time white noise on $\mathbb{R}^d$ - a random distribution locally in $B^{-(d+2)/2-\kappa}_{\infty,\infty}$
- Schauder estimates - gain of 2 degrees of regularity

Elliptic $d=4$ / Parabolic $d=2$

- $\xi\in B^{-2-\kappa}_{\infty,\infty} \quad \Rightarrow \quad \varphi\in B^{-\kappa}_{\infty,\infty}$

Elliptic $d=5$ / Parabolic $d=3$

- $\xi\in B^{-5/2-\kappa}_{\infty,\infty} \quad \Rightarrow \quad \varphi\in B^{-1/2-\kappa}_{\infty,\infty}$

- multiplication: $f\in B^{\alpha}_{\infty,\infty}$, $h\in B^{\beta}_{\infty,\infty}$ - the product fh well defined if $\alpha+\beta>0$

- renormalization needed: $\varphi^3\mapsto\varphi^3-\infty\varphi$

Parabolic local theory – $\mathbb{T}^d$

- $d=2$ – Da Prato–Debussche '03 – local solutions
- $d=3$ – Hairer '14 – local solutions by regularity structures
- $d=3$ – Catellier–Chouk '14 – local solutions by paracontrolled distributions
- $d=3$ – Mourrat–Weber '16 – coming down from infinity

$$\mathbb{E}\left[\sup_{0<t\leqslant 1}\sup_{\varphi_0\in B^{-1/2-\kappa}_{\infty,\infty}}\left(\sqrt{t}\|\varphi(t)\|_{B^{-1/2-\kappa}_{\infty,\infty}}\right)^p\right]<\infty$$

  - $B^{\alpha}_{p,q}$-spaces; L^p-energy estimates (testing by the $(p-1)^{\text{th}}$ power)

Parabolic global theory – $\mathbb{R}^d$

- $d=2$ – Mourrat–Weber '15 – global well-posedness
  - local solutions on $\mathbb{T}^2_M \Rightarrow$ global solutions on $\mathbb{T}^2_M \Rightarrow$ global solutions on $\mathbb{R}^2$

Our main results

Elliptic Φ^4 model on $\mathbb{R}^d, \ d=4,5$

- existence

Parabolic Φ^4 model on $\mathbb{R}_+ \times \mathbb{R}^d, \ d=2,3$

- existence, uniqueness, coming down from infinity

Ideas

- the theory is developed within the scale $B^{\alpha}_{\infty,\infty}$
- a new localization technique – splitting of distributions in weighted Besov spaces into
  - an irregular part which behaves nicely at the spatial infinity
  - a regular part that grows at infinity
- the use of maximum principle

- a smooth dyadic partition of unity $\sum_{k \geqslant -1} w_k = 1$ on $\mathbb{R}^d$

$$\mathcal{U}_{>} f := \sum_{k \geqslant -1} w_k \Delta_{> L_k} f \qquad \mathcal{U}_{\leqslant} f := \sum_{k \geqslant -1} w_k \Delta_{\leqslant L_k} f$$

where $\Delta_{> L_k} = \sum_{j : j > L_k} \Delta_j$ and $\Delta_{\leqslant L_k} = \sum_{j : j \leqslant L_k} \Delta_j$

- let $\rho(x) = \langle x \rangle^{-\nu} = (1 + |x|^2)^{-\nu/2}$ for some $\nu \geqslant 0$
- define $B^{\alpha}_{\infty,\infty}(\rho)$ by

$$\|f\|_{B^{\alpha}_{\infty,\infty}(\rho)} := \sup_{i \geqslant -1} 2^{i\alpha} \|\rho \Delta_i f\|_{L^\infty}$$

Lemma *Let $L > 0$ be given. There exists a (universal) choice of parameters $(L_k)_{k \geqslant -1}$ such that for all $\alpha, \delta, \kappa > 0$ and $a, b \geqslant 0$ it holds*

$$\|\mathcal{U}_{>} f\|_{B^{-\alpha-\delta}_{\infty,\infty}(\rho^{-a})} \lesssim 2^{-\delta L} \|f\|_{B^{-\alpha}_{\infty,\infty}(\rho^{-a+\delta})},$$

$$\|\mathcal{U}_{\leqslant} f\|_{B^{\kappa}_{\infty,\infty}(\rho^{b})} \lesssim 2^{(\alpha+\kappa)L} \|f\|_{B^{-\alpha}_{\infty,\infty}(\rho^{b-\alpha-\kappa})}.$$

Analysis of the elliptic Φ^4 model in $d = 4$

$$(-\Delta + \mu)\varphi + \varphi^3 - 3a\varphi - \xi = 0 \qquad \text{on} \quad \mathbb{R}^4$$

- a renormalization constant
- ansatz $\varphi = X + \phi + \psi$ where $(-\Delta + \mu)X = \xi$
- this gives

$$0 = (-\Delta + \mu)\phi + (-\Delta + \mu)\psi + [\![X^3]\!] + 3(\phi + \psi)[\![X^2]\!] + 3(\phi + \psi)^2 X + (\phi + \psi)^3$$

- we want to split

$$(-\Delta + \mu)\phi + \Phi = 0, \qquad (-\Delta + \mu)\psi + \psi^3 + \Psi = 0$$

where Φ contains all the irregular terms; Ψ all the regular terms

To treat the products:

- decompose in paraproducts and resonant term

$$(\phi+\psi)[\![X^2]\!] = (\phi+\psi) \prec [\![X^2]\!] + (\phi+\psi) \succ [\![X^2]\!] + (\phi+\psi) \circ [\![X^2]\!]$$

- include the localizers

$$(\phi+\psi) \prec [\![X^2]\!] = (\phi+\psi) \prec \mathcal{U}_{>}[\![X^2]\!] + (\phi+\psi) \prec \mathcal{U}_{\leqslant}[\![X^2]\!]$$

- leads to

$$(-\Delta+\mu)\phi + \Phi = 0, \qquad (-\Delta+\mu)\psi + \psi^3 + \Psi = 0$$

with

$$\Phi := [\![X^3]\!] + 3(\phi+\psi) \prec \mathcal{U}_{>}[\![X^2]\!] + 3(\phi+\psi)^2 \prec \mathcal{U}_{>} X$$

$$\Psi := \phi^3 + 3\psi\phi^2 + 3\psi^2\phi$$

$$+3(\phi+\psi) \prec \mathcal{U}_{\leqslant}[\![X^2]\!] + 3(\phi+\psi) \succcurlyeq [\![X^2]\!] \qquad +3(\phi+\psi)^2 \prec \mathcal{U}_{\leqslant} X + 3(\phi+\psi)^2 \succcurlyeq X$$

A bound for $\phi \in B^{\alpha}_{\infty,\infty}(\rho)$ for some $\alpha > 0$

- the stochastic objects can be constructed such that for $\sigma, \kappa > 0$

$$\|X\|_{B^{-\kappa}_{\infty,\infty}(\rho^{\sigma})}, \|[\![X^2]\!]\|_{B^{-\kappa}_{\infty,\infty}(\rho^{\sigma})}, \|[\![X^3]\!]\|_{B^{-\kappa}_{\infty,\infty}(\rho^{\sigma})} \lesssim 1$$

- we choose L for the localizers such that $\|\phi + \psi\|_{L^{\infty}(\rho)} \lesssim 2^{(2-\kappa-\alpha)L/2}$
- allows to control

$$\|\Phi\|_{B^{\alpha-2}_{\infty,\infty}(\rho)} = \|[\![X^3]\!] + 3(\phi + \psi) \prec \mathcal{U}_{>}[\![X^2]\!] + 3(\phi + \psi)^2 \prec \mathcal{U}_{>} X\|_{B^{\alpha-2}_{\infty,\infty}(\rho)} \lesssim 1$$

- a bound for $\phi \in B^{\alpha}_{\infty,\infty}(\rho)$ using Schauder estimates

A bound for $\psi \in B^{2+\beta}_{\infty,\infty}(\rho^{3+\beta}) \cap L^\infty(\rho)$ **for some** $\beta > 0$

Lemma *Let* ψ *be a classical solution. Then*

$$\|\psi\|_{B^{2+\beta}_{\infty,\infty}(\rho^{3+\beta})} \lesssim \|\Psi\|_{B^{\beta}_{\infty,\infty}(\rho^{3+\beta})} + \|\psi\|^{3+\beta}_{L^\infty(\rho)},$$

$$\|\psi\|_{L^\infty(\rho)} \lesssim 1 + \|\Psi\|^{1/3}_{L^\infty(\rho^3)}.$$

- Schauder estimate leads to

$$\|\psi\|_{B^{2+\beta}_{\infty,\infty}(\rho^{3+\beta})} \lesssim 1 + \|\psi\|^{3+\beta}_{L^\infty(\rho)},$$

- and then coercive estimate gives

$$\|\psi\|_{L^\infty(\rho)} \lesssim 1 + \|\psi\|^{1-\varepsilon}_{L^\infty(\rho)}$$

1. consider the problem on $\mathbb{T}^4_M$
2. show existence via Schaefer's fixed point theorem for the map

$$\mathcal{K}\colon B^{\beta}_{\infty,\infty}(\mathbb{T}^4_M) \times B^{\beta}_{\infty,\infty}(\mathbb{T}^4_M) \to B^{\beta}_{\infty,\infty}(\mathbb{T}^4_M) \times B^{\beta}_{\infty,\infty}(\mathbb{T}^4_M)$$

   where $\mathcal{K}(\tilde{\phi}, \tilde{\psi}) = (\phi, \psi)$ solves

$$(-\Delta + \mu)\phi + \Phi(\tilde{\phi}, \tilde{\psi}) = 0, \qquad (-\Delta + \mu)\psi + \psi^3 + \Psi(\tilde{\phi}, \tilde{\psi}) = 0$$

   (includes a variational proof of existence of the second equation)
3. pass to the limit $M \to \infty$ using the a priori estimates and compactness

Parabolic Φ^4 model in $d = 2, 3$

- regular initial conditions
- by-product of the elliptic a priori estimates
  - space-time polynomial weights
  - parabolic localizers: include a partition of unity in time
  - modified paracontrolled ansatz

- existence
  1. consider the equation driven by a mollified noise ξ_ε – solved by a classical theory
  2. decomposition + a priori estimates
  3. pass to the limit using the a priori estimates and compactness

- exponential weight of the form

$$\pi(t,x) = e^{-t\langle x\rangle^{2b}} \qquad b \in (0, 1/2)$$

- the equation for the difference of two solutions
- taking the advantage of our L^∞-bounds
- energy estimates in the L^2-scale: $\beta > 0$ small
  - $\partial_t \pi = -\pi \langle x\rangle^{2b}$ gives a good term on the LHS
  - the regular component in $L^\infty B^{\beta}_{2,2}(\pi) \cap L^2 B^{1+\beta}_{2,2}(\pi)$
  - the irregular one in $L^\infty B^{-\beta}_{2,2}(\pi) \cap L^2 B^{1-\beta}_{2,2}(\pi)$
  - Gronwall

- repeat the a priori estimates using the weight $\tau(t) = 1 - e^{-t}$
- new Schauder and coercive estimates
- modified paracontrolled ansatz
- we show

$$\phi \in CB^{\alpha}_{\infty,\infty}(\tau^{1/2}\rho) \cap CB^{1/2+\alpha}_{\infty,\infty}((\tau^{1/2}\rho)^{3/2+\alpha})$$

$$\psi \in CB^{2+\beta}_{\infty,\infty}((\tau^{1/2}\rho)^{3+\beta}) \cap L^{\infty}L^{\infty}(\tau^{1/2}\rho)$$

uniformly in the initial condition

Thank you for your attention!