Controllability properties and irreducibility of SPDEs driven by Lévy Processes

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The solution operator

- H denotes a separable Hilbert space
- $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ be a possible unbounded linear operator;
- $B : H \to H$ is a densely defined nonlinear operator taking values in H
- Let $\mathcal{U}$ be the space of controls, e.g. $\mathcal{U} = L^2(0, T; U)$.

Let u^c be a solution to the following equation

$$(*) \begin{cases} \dot{u}^c(t,x,v) &= \mathcal{L}u^c(t,x,v) + B(u^c(t,x,v)) + C \underbrace{v(t)}_{=\text{control}}, \quad t \geq 0, \\ u^c(0,x,v) &= x. \end{cases}$$

Fix $T > 0$. Let us denote by

$$\mathcal{R}_T : H \times L^2(0, T; U) \to H \tag{1}$$

the operator that takes each function $v \in L^2(0, T; U)$ and initial condition $x \in H$ to the solution $u^c(T, x, v)$ of the system $(*)$.

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- We say that the system $(*)$ is *exact controllable in time T*, iff for any $x \in H$

$$\mathcal{R}_T(x, L^2(0,T;U)) \supset H.$$

- We say that the system $(*)$ is *approximate controllable in time $T > 0$* iff for any $x \in H$

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Examples without nonlineartity

$B = 0$:

- The heat equation with Neumann boundary control is approximate controllable in time $T > 0$ (Laroche, Martin and Rouchon 2000, Coron (2007) Theorem 2.76);

- The wave equation with Neumann boundary control is exactly controllable in time $T > 2\pi$ (Zuazua 1991, Tucsnak and Weiss 2009);

Controllability Concepts and Stochastic Equations

Let M be a semimartingale and let u be a solution of

$$(**) \begin{cases} \dot{u}(t,x) &= \mathcal{L}u(t,x,v) + B(u(t,x)) + C\,\dot{\mathbf{M}}(\mathbf{t}), \quad t \geq 0, \\ u(0,x) &= x. \end{cases}$$

Definition:

Let $(\mathcal{P}_t)_{t\geq 0}$ be the Markov semigroup defined by

$$[\mathcal{P}_t\Phi](x) = \mathbb{E}[\Phi(u(t,x))], \; \Phi \in B_b(H), x \in H, \, t \geq 0,$$

where $u(\cdot,x)$ is the unique solution to $(**)$ with initial condition $x \in H$.

Definition

Given a Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$

- $(\mathcal{P}_t)_{t\geq 0}$ has the Feller property, iff $\mathcal{P}_t f \in C_\infty(H)$ for $f \in C_\infty(H)$ and $t > 0$.
- $(\mathcal{P}_t)_{t\geq 0}$ has the strong Feller property, iff $\mathcal{P}_t \mathcal{B}_b(H) \subset \mathcal{C}_b(H)$ for any $t > 0$.

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Controllability Concepts and properties of $(\mathcal{P}_t)_{t\geq 0}$

In many situations, replacing the noise by a control term gives information on the solution.

- strong Feller property
- support Theorem and irreducibility
- blow up of solutions (Debussche and de Bouard 2005)

Idea:

- By the Girsanov transform one generates a drift term mimicking the control;
- Small ball probabilities give the result.

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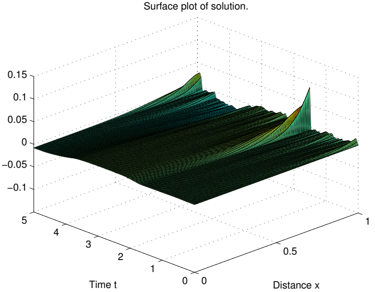
related Results

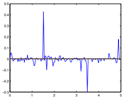
- Da Prato: controllability and strong Feller property;
- Röckner and Wang: controllability implies strong Feller but with nondegenerate Gaussian part.
- Zabczyk and Priola (2011): spectral noise (noise is a sum of independent Lévy processes multiplied by the eigenfunctions) support theorems.
- Maslowski;
- Hausenblas and Razafimanbimby (2015): approximate controllability and exact controllability implies reducibility
- Mattingly (talk yesterday), Glatt-Holtz, Geordie.

Controllability versus Irreducibility

The heat equation with Neuman boundary noise

We are interested in a model of a one dimensional rod, on side is perfectly isolated the other side is exposed to a fire.





Theorem

(H. and Razafimandimy (2015)) For any $x, y \in H$ and $\delta > 0$ there exists a $\kappa > 0$ such that

$$\mathbb{P}\left(u(T,x) \in \mathcal{D}_H(y,\delta)\right) \geq \kappa. \tag{3}$$

Controllability Concepts

Definition

A system is controllable in time $T > 0$ for a finite dimensional subspace $F \subset H$, iff for any $x \in H$

$$\pi_F \mathcal{R}_T(x, L^1(0, T; \mathcal{K})) \supset F.$$

Definition

A system is *solidly controllable* in time $T > 0$ for a finite dimensional subspace $F \subset H$, iff for any $R > 0$ and any $x \in H$, there exists an $\epsilon > 0$ and a compact set $K_\epsilon \subset L^1(0, T; \mathcal{K})$ such that for any function $\Phi : K_\epsilon \to F$ satisfying

$$\sup_{x \in K_\epsilon} |\Phi(x) - \pi_F \mathcal{R}_T(x, x)|_F \le \epsilon,$$

we have

$$\Phi(K_\epsilon) \supset B_F(R).$$

Controllability Concepts

Remark

(Agrachev and Sarychev (2005)) The 2D-Navier Stokes with set of controls

$$\mathcal{K} := \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(x_1), \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(x_1), \right.$$
$$\left. \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(x_1 + x_2), \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(x_1 + x_2) \right\}$$

described above is solid controllable on finite dimensional spaces.

Abstract Result

Let
- (U, d) be a metric space, X and F be finite-dimensional vector spaces;
- $f : U \times X \to F$ a continuous operator;
- for any probability measure $\mu \in P(X)$ and any $u \in U$ let us define

$$\mu_f(u, \cdot) : \mathcal{B}(X) \ni B \mapsto \mu(f^{-1}(u, B)) \in [0, 1].$$

Theorem

(Agrachev et. all) Suppose that,

- for any $u \in U$, the function $f(u, \cdot)$ is analytic and the interior of the set $f(u, X)$ is nonempty.
- $\mu \in \mathcal{P}(X)$ is absolutely continuous w.r.t. the Lebesgue measure on X.

Then,

- For any $x \in U$, the measure $\mu_f(x, \cdot)$ is absolutely continuous with respect to Leb_F;
- The function $\mu_f : U \ni u \mapsto \mu_f(u, \cdot) \in \mathcal{P}(F)$ endowed with the total variation norm[a] is continuous.

[a] The total variation distance between two probability measures P and Q on a probability space $(\Omega, \mathcal{F})$ is defined by $\delta_{var}(P, Q) = \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$.

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The proof is a straightforward application of the Sard Theorem.

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The image under an analytic function of the set of singular points has measure zero.

How to extend this result to infinite dimensional measures?

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Decomposability of a Measure

Definition

Let H be a Hilbert space. We call a measure μ decomposable, iff there exists an ONB $\{e_n : n \in \mathbb{N}\}$ in H such that

$$\mu = \otimes_{n=1}^{\infty} \mu_n,$$

and μ_n is defined by

$$\mathcal{B}(\mathbb{R}) \ni B \mapsto \mu_n(B) := \mu\left(\{h \in H : \langle h, e_n \rangle \in B\}\right), \ n \in \mathbb{N}.$$

Example

Let us consider the process $\{W(t) : t \in [0,1]\}$, where W is a Wiener process. Then, the Wiener measure is decomposable. In fact, one can find an ONB $\{e_n : n \in \mathbb{N}\}$ such that the random variables

$$\xi_n := \int_0^1 e_n(s) dW(s), \quad n \in \mathbb{N},$$

are independent.

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Let
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- for any probability measure $\mu \in P(X)$ and any $u \in U$ let us define

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Theorem

(Agrachev et. all) Let us assume that

- $\forall u \in H$ $f(u, \cdot)$ is analytic and f has continuous Frechet derivative with resp. to (u, x)
- $\forall u \in U$, $\exists$ finite dimensional subspace $\tilde{X}_u \subset X$ and a ball $B_u \subset \tilde{X}_u$ such that the interior of the set $f(u, B_u)$ is non-empty.

Let $\mu \in \mathcal{P}(X)$ being decomposable and with finite second moments. Then

- For any $x \in U$, the measure $\mu_f(x, \cdot)$ is absolutely continuous w.r.t. Leb_F;
- The function $\mu_f : U \ni u \mapsto \mu_f(u, \cdot) \in \mathcal{P}(F)$ endowed with the total variation norm is continuous.

Application of the Abstract result

Given the SPDE

$$(\star)\begin{cases} \dot{u}(t,x) &=& \mathcal{L}u(t,x,v) + B(u(t,x)) + W(t), \quad t \geq 0, \\ u(0,x) &=& x. \end{cases}$$

Theorem

(Agrachev, Kuksin, Sarychev, and Shirikyan, 2007) Given the control system corresponding to $(\star)$ is solid controllable. Then, for any finite dimensional subspace F of H, the measure

$$\mathcal{B}(F) \ni U \mapsto \mathbb{E}1_U(\pi_F u(T,x)),$$

where π_F denotes the orthogonal projection onto F, has a absolutely continuous density with respect to the Lebesgue measure. In addition, the measure

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Ingredients of the proof

- solid controllability (see Agrachev and Sarychev (2005));
- analyticity of the solution operators $\mathcal{R}_T$ (see Kuksin (1998));
- decomposability of the Wiener measure.

The stochastic Navier Stokes with Lévy noise

Then the initial-boundary value problem associated with the 2D– Navier–Stokes equations is given by

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} & - & \nu \triangle u + (u \cdot \nabla) u + \nabla p = \dot{L}_t \quad \text{in } \mathbb{T}^2, \\ \operatorname{div} u & = & 0 \text{ in } \mathbb{T}^2, \\ u(\cdot, 0) & = & x \text{ in } \mathbb{T}^2, \end{array} \right. \tag{4}$$

with

$$\begin{aligned} L_t &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(x_1) l_t^1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(x_1) l_t^2 \\ &\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(x_1 + x_2) l_t^3 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(x_1 + x_2) l_t^4. \end{aligned}$$

Here, $l = \{l_t^j : t \geq 0\}$, $j = 1, \ldots, 4$, are independent, one dimensional, α–stable processes. Uniqueness and Existence of a solution is given by Brzezniak, H. and Jia Hui (2014).

Irreducibility Results

Theorem

(H. and Razafimandimby 2017) Let u be the solution of the stochastic Navier Stokes driven by a Lévy processes described above.
Then, for any finite dimensional subspace F of H, the measure

$$\mathcal{B}(F) \ni U \mapsto \mathbb{E}1_U(\pi_F u(T,x)),$$

where π_F denotes the orthogonal projection onto F, has a absolutely continuous density with respect to the Lebesgue measure. In addition, the measure

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Why is the decomposability of the measure is important?

Notation

- E be a separable Banach space, $\mathcal{B}(E)$ be the σ–algebra generated by the open sets and μ be a probability measure on $(E, \mathcal{B}(E))$;
- Let $E = F_n \oplus G^n$ be the algebraic direct sum, where
  - F_n is finite dimensional given;
  - G^n is infinite and the complement;
- let $\mu_{(F_n,G^n)}$ the probability measure defined by

$$\mu_{(F_n,G^n)} : \mathcal{B}(F_n) \ni A \mapsto \mu(A + G^n) \in [0, 1];$$

- For $A \subset E$ and $y \in G^n$ let $A_{(F_n,G^n)}(y) = \{x \in F_n : x + y \in A\}$;

What we need:

For all $n \in \mathbb{N}$ there exists a kernel

$$m_n : G^n \times \mathcal{B}(F_n) \to \mathbb{R}_0^+$$

such that $\mu(A) = \int_{G^n} \int_{A_{(F_n,G^n)(y)}} m_n(y, dx) \mu_{G^n}(dy).$

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Corollary

Let E be a separable Banach space and $\{e_n : n \in \mathbb{N}\}$ be a Schauder basis. Put $E_n := \{\lambda e_n : \lambda \in \mathbb{R}\}$ and $F_n := E_1 \oplus \cdots \oplus E_n$, $\mu_n = \Pi_{E_n}\mu$.

- for all $n \in \mathbb{N}$ μ_n is absolutely continuous with respect to the Leb_{F_n}
- for any $y \in F_{n-1}$ the probability measure

$$\mathcal{B}(\mathbb{R}) \ni A \mapsto \mathbb{P}\left(\{x \in E : \langle x, e_n\rangle \in A\} \mid \pi_{F_{n-1}} = y\right)$$

  is absolutely continuous.

Then for any $n \in \mathbb{N}$ there exists a function $h_n : G^n \times E_1 \oplus \cdots \oplus E_n \to \mathbb{R}_0^+$ such that μ_n–a.s.

$$m_n(y, A) = \int_A h_n(y, x)\mu_n(dx).$$

Decomposition by Debauchies wavelets

Let ψ be the mother wavelet and ϕ the to ψ associated scaling function given
The wavelet system is given by

$$\psi_{j,k} := 2^{-\frac{j}{2}}\psi(2^j t + k) \text{ and } \phi_{j,k} := 2^{-\frac{j}{2}}\phi(2^j t + k), \quad k \in \mathbb{N}, j = 0, \ldots, 2^k - 1.$$

The corresponding multiresolution analysis is defined by

$$V_n := \text{span}\{\phi_{j,k} : j = 1, \ldots, n,\ j = 1, \ldots, 2^j\},$$

and

$$W_n := \text{span}\{\psi_{n,k} : j = 1, \ldots, 2^n\}.$$

Note, that

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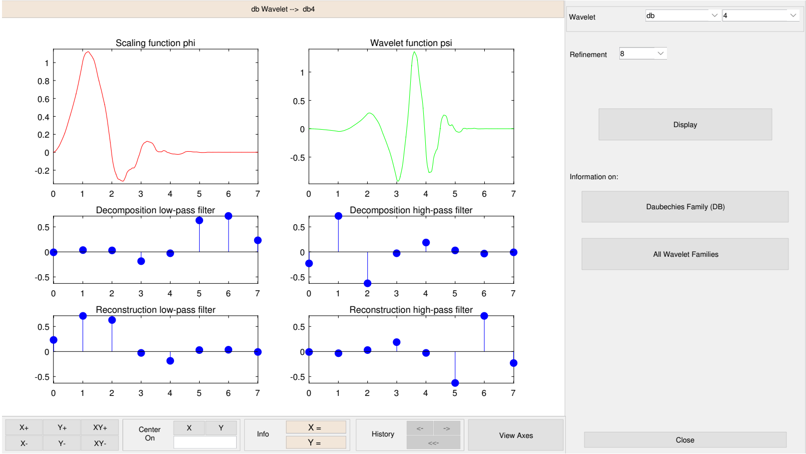

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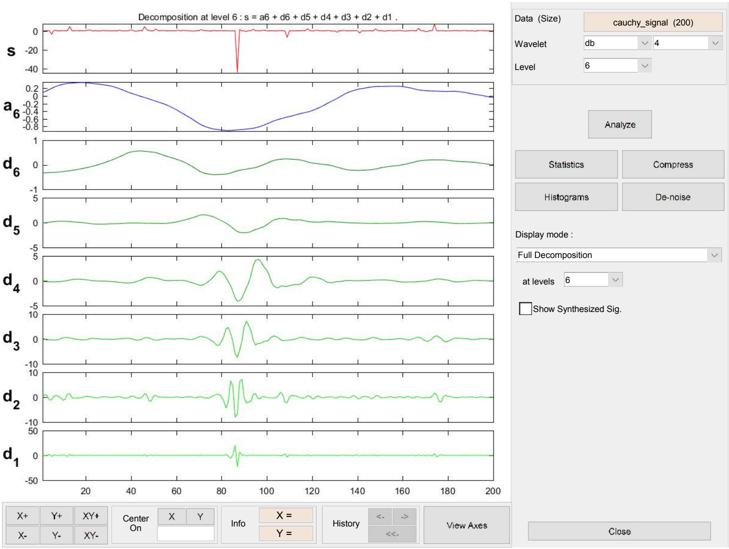

Decomposition by Daubechie wavelets

Theorem

Let $0 < p < \infty$, $0 < q < \infty$, $s \in \mathbb{R}$ and the mother wavelet and scaling function be u times continuously differentiable with $u > \max(s; 1 - \frac{1}{p} - s)$. Then the wavelet system is an unconditional basis in $B^s_{p,q}(\mathbb{R})$. (See Triebel or Kahane and Lemarié-Rieusset)

A compound Poisson process with intensity ν can be written

$$\int_0^1 \int_{\mathbb{R}} z\eta(dz, ds) = \sum_{i=1}^N \delta_{T_i} Y_i,$$

where

- N is Poisson distributed with parameter $\nu(\mathbb{R})$;
- $\{T_i : i \in \mathbb{N}\}$ independent and uniform distributed in $[0, 1]$;
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Note, that $\delta \in B^s_{p,p}(\mathbb{R})$ for $s < \frac{1}{p} - 1$.

Decomposition by Daubechie wavelets

Theorem

Let $0 < p < \infty$, $0 < q < \infty$, $s \in \mathbb{R}$ and the mother wavelet and scaling function be u times continuously differentiable with $u > \max(s; 1 - \frac{1}{p} - s)$. Then the wavelet system is an unconditional basis in $B^s_{p,q}(\mathbb{R})$. (See Triebel or Kahane and Lemarié-Rieusset)

A compound Poisson process with intensity ν can be written

$$\int_0^1 \int_{\mathbb{R}} z\eta(dz, ds) = \sum_{i=1}^N \delta_{T_i} Y_i,$$

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Decomposition by Daubechie wavelets

Let η be a Poisson random measure on $\mathbb{R}$ with intensity ν. Let us define for a continuous function $f : [0,1] \to \mathbb{R}$

$$\xi(f) := \int_0^1 \int_{\mathbb{R}} f(s)\eta(dz, ds),$$

and $\psi_{j,k}(f) = \int_0^1 \psi_{j,k}(s)\, f(s)\, ds$ and $\phi(f) = \int_0^1 f(s)\phi(s)\, ds$.
Note, that

$$\xi(f) := \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \zeta_{j,k}\psi_{j,k}(f) + a_0\phi(f), \quad t \in [0,1].$$

where $\{\zeta_{j,i} : j \in \mathbb{N}, i = 1, \ldots, 2^j\}$ is a family of random variables, such that

$$\zeta_{j,k} \overset{d}{=} \int_0^1 \int_{\mathbb{R}} \psi_{j,k}(s)\, z\eta(dz, ds),$$

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Decomposition by Daubechie wavelets

Proposition

Let us assume for $p \in [1,2)$ and $\int |z|^p \nu(dz) < \infty$. Then, for $s < \frac{1}{p} - 1$, we have

$$\mathbb{E}|\xi|^p_{B^s_{p,p}} < \infty.$$

Proposition

The probability

$$\mathcal{B}(\mathbb{R}) \ni U \mapsto \mathbb{P}\left(\zeta_{n,k} \in U\right)$$

is absolutely continuous with respect to the Lebesgue measure.

Proposition

The conditional probability

$$\mathcal{B}(\mathbb{R}) \ni U \mapsto m^n_k(U,x) := \mathbb{P}\left(\zeta_{n,k} \in U \mid \mathcal{F}(V_n)\right)$$

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For a function $f : [0,1] \to \mathbb{R}$ we write

$$\xi(f) := \int_0^1 \int_{\mathbb{R}} f(s) z \, \eta(dz, ds).$$

Lemma

Let $f : [0,1] \to \mathbb{R}$ such that there exists a $\delta > 0$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$ such that $|f(s)| \geq \delta$ for all $s \in [t_1, t_2]$. Then

1. $\mathrm{supp}(\xi(f)) = \mathbb{R}$;
2. the law of $\xi(f)$ is absolutely continuous with respect to the Lebesgue measure.

The proof follows by the fact that a Levy process with infinite intensity has absolutely continuous distribution and support $\mathbb{R}$ (Sato, Chapter 27).

Small deviation property

Proposition

For $s < \frac{1}{p} - 1$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}\left(|\xi|_{B^s_{p,p}} \leq \epsilon \right) \geq \delta.$$

This follows from a result of Derreich and Arzuda.

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The same result can be shown by the stochastic Navier Stokes driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

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Same result should be possible for shell models.

Possible Extension to infinite dimension

Lie–Trotter splitting of the Markovian semigroup in low modes and high modes $\Rightarrow$ determining modes.

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Thank you for your attention