Quasilinear singular SPDEs

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General aim

Solution theory for quasilinear SPDEs:

$$\partial_t u = a(u) \bullet \Delta u + \sum_{j=0}^{2} F_j(u) \bullet (\nabla u)^{\otimes j} + G(u) \bullet \xi \,, \qquad \text{(SPDE)}$$

with ξ a generalised Gaussian random field and $u(t,\cdot)\colon \mathbf{T}^d \to \mathbf{R}^m$.
Recall $\xi \in \mathcal{C}^{\alpha-2}$ if $|\mathrm{Cov}(z)| \lesssim |z|^{-\beta}$ for $\beta < 4 - 2\alpha$.

Problem: All red products typically ill-defined if $\xi \in \mathcal{C}^{\alpha-2}$ for $\alpha < 1$. Blue product ill-defined for $j = 2$, $\alpha < \frac{2}{3}$ and $j = 1$, $\alpha < \frac{1}{2}$. Interesting case of space-time white noise in $d = 1$ corresponds to $\alpha < \frac{1}{2}$.

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Semilinear case

One can "guess" the local behaviour of solutions:

$$\partial_t u = \Delta u + F(u)(\partial_x u)^2 + G(u)\xi \; .$$

For (y, s) near (x, t), one would expect

$$u(y, s) \approx u(x, t) + G(u(x, t))\big(v(y, s) - v(x, t)\big) \; ,$$

with v solving $\partial_t v = \Delta v + \xi$.

Hope: Maybe defining the various products for v instead of u already yields enough information? Should be easier since v is somewhat explicit...

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In our case

Graphical notations for symbols: ∘ for noise, — for heat kernel, **—** for derivative of heat kernel. For example $(\Pi_{z_0}\,)(z) = v(z) - v(z_0)$.
(! Suitable recentering required !) One can guess the expansion

$$U = u\,\mathbf{1} + G + GG' + FG^2 + u'\,X + 2FG^2G' + 2F^2G^3 + G^3F' + \frac{1}{2}G^2G'' + G(G')^2 + FG^2G' + u'G' + 2FGu' .$$

Remark: Similar to expansions in Feynman diagrams. However, coefficients are not constant but depend on the solution itself. Also, finitely many terms suffice!

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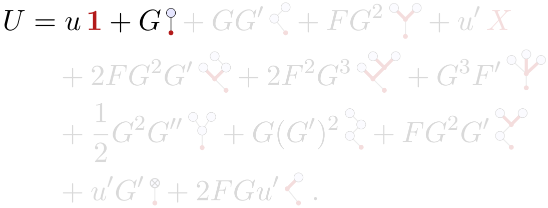

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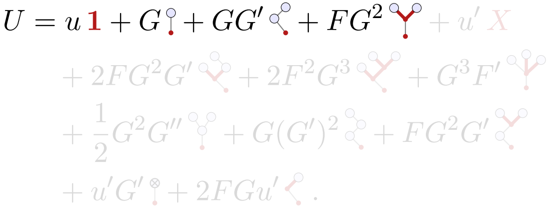

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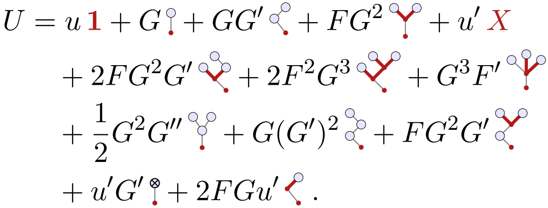

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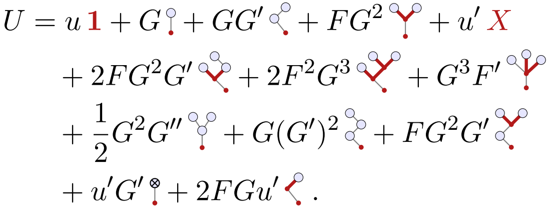

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General methodology

Perform following steps, with $\mathcal{T}$ space of linear combinations of symbols.

1. Find suitable $\mathcal{T}$ and define spaces $\mathcal{D}^\gamma$ of fcns $\mathbf{R}^{d+1} \to \mathcal{T}$. Definition depends on "model" Π.
2. Define all operations required to reformulate problem as fixed point in $\mathcal{D}^\gamma$ with unique solutions.
3. Show that, modulo renormalisation, models for ξ^ε converge to a limit independent of choice of regularisation procedure.
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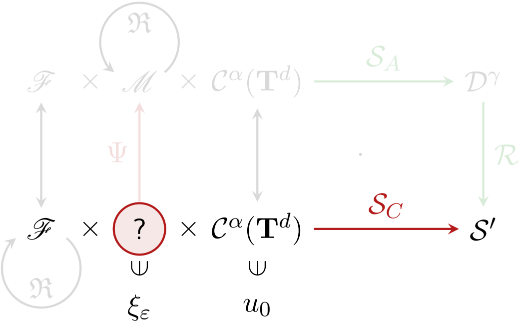
General picture

Pictorial representation of method:



$\mathscr{F}$: Possible right-hand sides for the equation.

$\mathcal{S}_C$: Classical solution to the PDE with smooth input.

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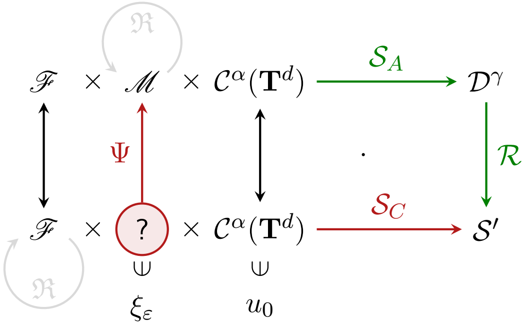

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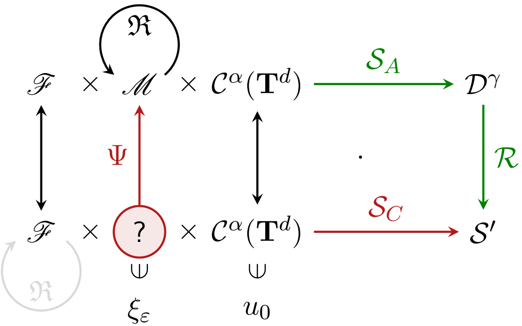

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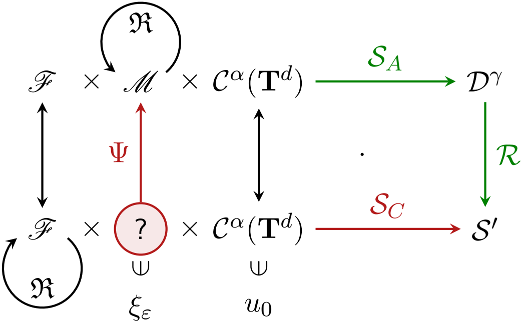

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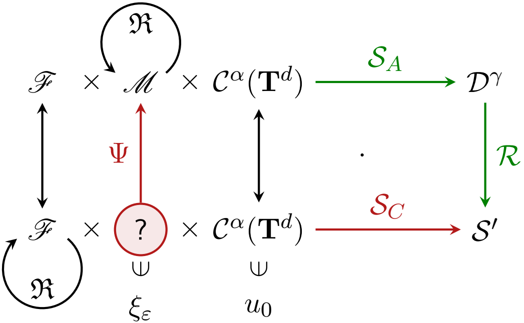

Probabilistic step: find $M_\varepsilon \in \mathfrak{R}$ such that $M_\varepsilon \Psi(\xi_\varepsilon)$ converges in probability.

Back to quasilinear case

Consider simplest non-trivial case:

$$\partial_t u = a(u)\Delta u + G(u)\xi \ .$$

This time, for (y, s) near (x, t), one would expect to lowest order

$$u(y, s) \approx u(x, t) + G(u(x, t))\big(v_{a(u(x,t))}(y, s) - v_{a(u(x,t))}(x, t)\big) \ ,$$

with v_a solving $\partial_t v_a = a\Delta v_a + \xi$.

Natural Idea: Formulate this as $U = u\,\mathbf{1} + G(u)\,\delta_{a(u)} \otimes$.

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Augmented model space

In semilinear case, each symbol τ generates a one-dimensional subspace $\langle\tau\rangle$ of $\mathcal{T}$. In quasilinear case, we postulate that it generates a subspace of the form $S_k \otimes \langle\tau\rangle$ with S_k a space of distributions in k variables, where k is the number of ‘edges’ of τ.

Example: for $\tau =$, one has $k = 2$ and

$$\Pi(\eta \otimes \text{[tree]}) = \int \big(P'(a_1,\cdot) \star \xi\big)\big(P'(a_2,\cdot) \star \xi\big)\, \eta(da_1, da_2)\ .$$

Here, $P(a,\cdot)$ is heat kernel for $\partial_t - a\Delta$.

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Fixed point problem

Freezing of coefficients: define operators

$$I_\ell^{(k)}(b,f)(z) = \int (D_1^\ell D_2^k P)(b(z), z-z') f(z')\,dz'\;.$$

Lemma: PDE $\partial_t u = a(u)\Delta u + f$ is equivalent to

$$\begin{aligned} u &= I(a(u),\hat f)\;,\\ \hat f &= (1 - a'(u) I_1(a(u),\hat f)) f + (aa'')(u)(\partial_x u)^2 I_1(a(u),\hat f)\\ &\quad + (a(a')^2)(u)(\partial_x u)^2 I_2(a(u),\hat f) + 2(aa')(u)\partial_x u\, I_1^{(1)}(a(u),\hat f)\;. \end{aligned}$$

'Nice' fixed point problem for pair $(u,\hat f)$.

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Remaining steps

Steps required to build full solution theory:

1. Build 'abstract version' of operators $I_\ell^{(k)}$ and show that they map $\mathcal{D}^\gamma$ into $\mathcal{D}^{\gamma+2-|k|}$. (Easy)
2. Show that the FP problem admits unique local solutions for every model. (Straightforward; works for all $\alpha > 0$!)
3. Show convergence of renormalised 'augmented' models. (Follows from existing result with A. Chandra.)
4. Show that renormalisation generates local counterterms. (Lengthy calculation, done only in the case $\alpha > \frac{1}{2}$.)

Alternative approach (Otto-Weber): build more 'direct' solution map to $\partial_t u = a(u)\Delta u + f$ in similar augmented reg. structure framework.

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Sample 'classical' result

Consider

$$\partial_t u = a(u)\Delta u + F(u)\,(\partial_x u)^2 + H(u) + G(u)\xi_\varepsilon\ . \qquad (\star)$$

with ξ_ε an ε-regularisation of ξ with $\mathrm{Cov}(z) \sim |z|^{2\alpha-4}$ for $\frac{1}{2} < \alpha < 1$.

Theorem: For every mollifier ϱ, there exists a (essentially unique) function $a \mapsto c(a)$ (depending on ϱ) such that if

$$H(u) = H_0(u) + \frac{c(a)}{\varepsilon^{2-2\alpha}}\big((aG'G)(u) + (G^2F)(u) - (a'G^2)(u)\big)$$

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- Itô isometry in white noise case?
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