Path-by-path regularization by noise for scalar conservation laws

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joint work with: Panagiotis E. Souganidis, Benoit Perthame, Paul Gassiat, Mario Maurelli, Khalil Chouk
[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2017], [G., Perthame, Souganidis; SINUM, 2016], [Gassiat, G.; PTRF, 2018+], [Maurelli, G.; arXiv:1701.05393], [Chouk, G.; arXiv:1512.06056].

Outline

Introduction

- Aim: Improved regularity and well-posedness for PDE by noise, in particular in fluid dynamics, e.g. $3d$-Navier-Stokes equations.

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- More recent: Linear multiplicative noise

$du + b(x) \cdot \nabla u \, dt = \nabla u \circ d\beta_t$ [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010].

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- Side-remark: In certain cases [Delarue, Flandoli, Vincenzi; CPAM, 2014] space-time linear multiplicative noise

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- Different forms of noise?

Regularization by nonlinear noise

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Regularity of solutions for stochastic SCL

- Consider mean field equations

$$dX_t^i = \sigma^L\left(X_t^i, \frac{1}{L}\sum_{j=1}^{L}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^d$$

Taking $L\to\infty$ and $\sigma^L\to\sigma$ leads to stochastic scalar conservation laws

$$du+\operatorname{div}(\underbrace{\sigma(x,u)u}_{=:A(x,u)}\circ d\beta)=0 \quad \text{on } (0,T)\times\mathbb{R}^d.$$

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- Methods apply to general spatially homogeneous and truly nonlinear flux A, general dimension.
- For simplicity, in this talk restrict to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0.$$

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}^d$$
$$u(0) = u_0 \in L^\infty(\mathbb{R}^d).$$

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For

$$\chi(t,x,v) = \chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$$

we get the kinetic form

$$\partial_t\chi + v\partial_x\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}.$$

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Dissipation-dispersion approximations lead to

Definition (De Lellis, Otto, Westdickenberg, 2003)

A function $u \in L^\infty([0,T]\times\mathbb{R}^d)$ is said to be a quasi-solution if $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t\chi + v\partial_x\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}^d\times\mathbb{R}$$

for some finite (signed) measure m.

Theorem (De Lellis, Westdickenberg, 2003; Jabin, Perthame 2002)

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Then

1. *Each quasi-solution satisfies, for all* $\lambda \in (0, \frac{1}{3})$,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

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2. *For each* $\lambda > \frac{1}{3}$ *there exists a quasi-solution* u *such that*

$$u \notin L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

Theorem (G., Souganidis; CPAM, 2016)

Let u be a quasi-solution to

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } (0,T)\times\mathbb{T}.$$

Then,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{T})) \quad \text{for all } \lambda \in (0,\frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

If u is an entropy solution, then

$$u(t) \in W^{\lambda,1}(\mathbb{T}) \quad \text{for all } t>0, \lambda \in (0,\frac{1}{2}), \mathbb{P}\text{-a.s.}. \qquad (\star)$$

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Two resulting questions:

1. Can the zero set in $(\star)$ be chosen uniformly in t?
2. Characterize the properties of Brownian paths leading to $(\star)$.

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Framework

- Model example:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } (0,T)\times\mathbb{T},$$

with $w \in C([0,T];\mathbb{R})$.

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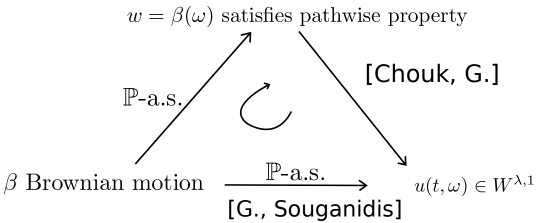
- Again: Results are given for general truely nonlinear flux A.
- How to classify pathwise properties of w leading to improved regularity?



Idea of the proof

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- Kinetic formulation:

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  for some finite Radon measure m.
- Change of variables gives

$$\chi(t,x,v) = \chi_0(x + v\beta_t, v) + \int_0^t \partial_v m(s, x + v(\beta_t - \beta_s), v)ds.$$

Idea of the proof

- Averaging over velocity

$$u(t,x) = \int_v \chi = \int_v \chi_0(x + v\beta_t, v)dv + \int_0^t \int_v \partial_v m(s, x + v(\beta_t - \beta_s), v)dvds.$$

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- Rigorously, this can be seen by Fourier transform, that is,

$$\hat{u}(t,n) = \int_v e^{-iv\beta_t n}\hat{\chi}_0(n,v)dv + \int_0^t \int_v e^{-iv(\beta_t - \beta_s)n}\partial_v \hat{m}(s,n,v)dvds.$$

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- The oscillatory integrals have a regularizing effect, both in v and in $\beta_t - \beta_s$.

Framework

- For SDE this has been considered by [Catellier, Gubinelli; *SPA*, 2016]: A path $w \in C(\mathbb{R}_+;\mathbb{R}^d)$ is said to be (ρ,γ)-irregular if

$$|\int_s^t e^{iw_r \cdot n} dr| \lesssim (1+|n|)^{-\rho}|t-s|^\gamma \quad \forall n \in \mathbb{R}^d, s < t.$$

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- Note:

$$\int_s^t e^{iw_r \cdot n} dr = \int_{\mathbb{R}} e^{ix\cdot n} dL_w^{s,t}(x) = \hat{L_w^{s,t}}(n)$$

the Fourier transform of the local time.

Main result

Theorem

Let $w \in C^\eta([0,T],\mathbb{R}^d)$ for some $\eta > 0$ be (ρ,γ)-irregular, u a quasi-solution solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } (0,T)\times\mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta+1)-(1-\gamma)}{(\rho\vee 1)(\eta+1)+(1-\gamma)},$$

we have

$$u \in L^1([0,T];W^{\lambda,1}(\mathbb{T})).$$

Corollary

Let β^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$ and u be a quasi-solution to

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Then, $\mathbb{P}$-a.s. for all $\lambda < \frac{1}{1+2H}$,

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- Note: Fully recover the probabilistic result from [G., Souganidis; *CPAM*, 2016]: For $H = \frac{1}{2}$ get $\lambda < \frac{1}{2}$.

A path-by-path scaling condition

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Discussion of the path classification

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- However: (ρ,γ)-irregularity depends on two parameters, also encoding a time regularity. Hence, does not seem to be optimal.
- Moreover: (ρ,γ)-irregularity not easy to check.
- To avoid the use of oscillatory integrals: Completely avoid Fourier methods in the proof.

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- Rewrite as, for $\lambda > 0$,

$$\partial_t \chi + v\partial_x \chi \circ dw_t + \lambda \chi = \partial_v m + \lambda \chi.$$

Idea of the proof

- Change of variables, take velocity integral and drop i.c.:

$$u(t,x) = \int_v \chi(t,x,v)\,dv = \int_0^t \int_v e^{-\lambda(t-s)}(\partial_v m)(s,x-vw_{s,t},v)\,dvds$$
$$+\lambda \int_0^t \int_v e^{-\lambda(t-s)}\chi(s,x-vw_{s,t},v)\,dvds.$$

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- Introduce the random X-ray transform

$$(Tg)(t,x) := \int_0^t \int_v g(s,x-vw_{s,t},v)e^{-\lambda(t-s)}\,dvds.$$

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where m is a finite measure and $\chi(t,x,v) := 1_{[0,u(t,x)]}(v)$.

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where m is a finite measure and $\chi(t,x,v) := 1_{[0,u(t,x)]}(v)$.

- Strategy: Estimate the regularity of $T(\partial_v m)$, $T\chi$ then use real interpolation.

Path-by-path scaling condition

- This leads to: *Path-by-path scaling condition*: Assume that there is a $\iota \in (0,1]$ such that for every $\sigma \in [0,1)$, $\lambda \geq 1$ we have

$$\int_0^T \int_0^{T-r} e^{-\lambda t} |\underbrace{w_{t+r} - w_r}_{=: w_{r,r+t}}|^{-\sigma} \, dt dr \lesssim \lambda^{-1+\iota\sigma}.$$

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- Easy to see: (ρ, γ)-irregularity implies path-by-path scaling.

Theorem

Let u be a quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } \mathbb{R}$$

and suppose that $w \in C^\eta$ *satisfies path-by-path scaling. Then, for all* $\lambda < \frac{1+\eta-\iota}{1+\eta+\iota}$,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{T})).$$