Stochastic solutions of 2D fluids

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Stochastic Partial Differential Equations, Luminy 2018

Literature

- F. Flandoli, Weak vorticity formulation of 2D Euler equations with white noise initial condition, to appear on Comm. PDEs

It deals with 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ in vorticity form

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

with random initial condition

$$\omega|_{t=0} = \text{white noise (in space)}$$

Revisits the following classical work, adding point vortex and smooth solution approximation:
- S. Albeverio, A. B. Cruzeiro, Global flows with invariant (Gibbs) measures for Euler and Navier–Stokes two-dimensional fluids, Comm. Math. Phys. 1990

Literature

- F. Flandoli, D. Luo, ρ-white noise solution to 2D stochastic Euler equations, arxiv

It deals with the stochastic 2D Euler equations

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega + \sum_{k \in \mathbb{Z}^2} \sigma_k(x) \cdot \nabla \omega \circ \beta_k'(t) &= 0 \\ \omega|_{t=0} &= \text{white noise} \end{aligned}$$

and the proof of a gradient type estimate for the associated Fokker-Planck equation

$$\langle \rho_t, F_t \rangle - \langle \rho_0, F_0 \rangle = \int_0^t \left(\int_{H^{-1-}} \rho_s \left(\mathcal{L} F_s + \partial_s F_s \right) d\mu \right) ds$$

$$\mathcal{L} f = \langle b(\omega), Df \rangle + \frac{1}{2} \sum_{k=1}^{\infty} \langle \sigma_k \cdot \nabla \omega, D \langle \sigma_k \cdot \nabla \omega, Df \rangle \rangle$$

Perspective of the talk

The above ones are rigorous results on 2D Euler equations with white noise initial conditions, motivated by questions of existence and uniqueness and the similarity with dispersive equations.

The perspective of the talk will be different, more heuristic but hopefully inspiring: a discussion of *2D turbulence* and why the previous models may have something to do with it.

Inverse cascade

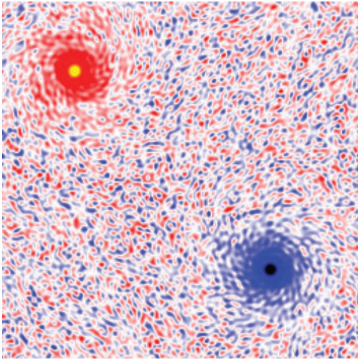
Inverse cascade means that some mechanism generates motion at very small scale and, for deep reasons partially not understood, larger scale motions are activated spontaneously, and energy is transferred from the smaller to the larger scales.

An example could be, idealizing as a two-dimensional flow the dynamics of the ocean surface layer, a laminar flow colliding with a cluster of islands. Permanent generation of small vortices is observed around the islands and some form of stationary inverse cascade is observed past the cluster.

Decaying inverse cascade

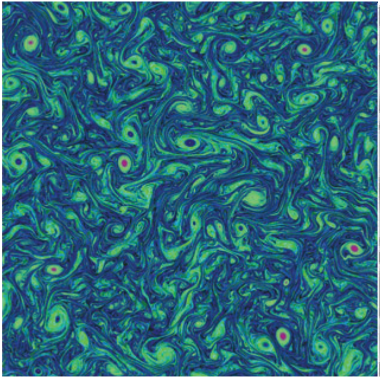
Decaying inverse cascade is when friction forces, which always act at large scales in 2D fluids, dissipate energy, but no new energy is injected at small scales.

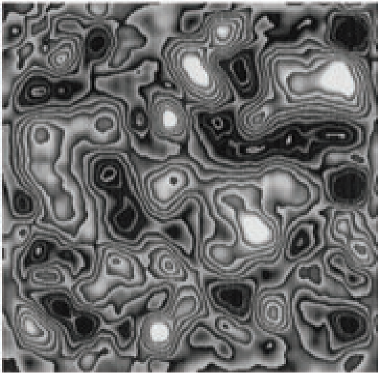
Thus the system goes at rest. Experimentally, it develops very coherent large scale structures. Primarily, we do not try to describe this case.



Stationary inverse cascade

Stationary inverse cascade is when the mechanism of generation at small scales is permanent and compensates the energy loss at large scales due to friction.





Modelling inverse cascade

In general terms, one should use the equation (we write the vorticity form)

$$\partial_t \omega + u \cdot \nabla \omega = -\lambda \omega + \textit{input at small scale}.$$

We neglect viscous dissipation which presumably is not relevant in this particular fluid configuration.

We do not start from general solutions to this equation but from a very particular one based on point vortices. We generate point vortices at random in space and time; the existing vortices move according to the classical dynamics of interacting point vortices; energy dissipation by friction is incorporated in a rule of decay of vortex intensity.

Point vortex model

The ingredients of the model are:

- N = number of point vortices at time $t = 0$; it is the parameter that we send to infinity
- $N(t)$ = number of point vortices at time t; new vortices are generated at random, no vortex disappear, hence $N(t)$ is a non-decreasing random function
- $X_i(t)$ = position at time t of vortex i in the space domain $D \subset \mathbb{R}^2$; vortex i exists on a time interval $[T_i, \infty)$
- $a_i^N(t)$ = intensity (circulation) at time t of vortex i
- λ = friction coefficient
- $K(x, y)$, $x, y \in D$, Biot-Savart kernel (interaction kernel between vortices).

Equations for the time evolution

The law of motion of vortex i, when it is alive, is

$$\frac{d}{dt}X_i(t) = F_i(t) := \sum_{j=1}^{N(t)} a_j^N(t)\, K\left(X_i(t) - X_j(t)\right).$$

The law of decay of the intensities is

$$\frac{d}{dt}a_i^N = -\lambda a_i^N.$$

Statistics of new vortices

Given N, it remains to describe the statistics of the triples

$$T_i^N, \qquad X_i := X_i|_{t=T_i^N}, \qquad a_i^N := a_i^N|_{t=T_i^N} :$$

- $T_i^N = 0$ for $i = 1, ..., N$
- $T_{N+j}^N = S_1^N + ... + S_j^N$
- $\left\{S_i^N, X_i^N, a_i^N; i \in \mathbb{N}\right\}$ all independent
- $S_i^N \sim Exp\left(\lambda_N\right)$ (hence $N\left(t\right) = N + \mathcal{N}_N\left(t\right)$, where $\mathcal{N}_N\left(t\right)$ is a Poisson process with intensity λ_N)
- $a_i^N = \epsilon_N Z_i$ with Z_i standard Gaussians
- X_i^N uniformly distributed on $\mathbb{T}^2$.

Two parameters: λ_N=average number new vortices, ϵ_N=vortex intensity.

Equation for the vorticity field

The measure-valued vorticity field

$$\omega^N(t) = \sum_{i=1}^{N(t)} a_i^N(t)\, \delta_{X_i^N(t)}$$

satisfies 2D Euler equations in weak form

$$\partial_t \omega^N + u^N \cdot \nabla \omega^N = -\lambda \omega^N + \underbrace{\sum_{i=1}^{\infty} \delta\left(t - T_i^N\right) a_i^N \delta\left(x - X_i^N\right)}_{\text{input at small scale}}$$

Scaling

One can analyze different limit regimes (they differ by the choice of λ_N, ϵ_N). Here we choose:

$$\begin{aligned}\lambda_N &= \left(1-e^{-\lambda}\right) N \overset{\lambda \text{ small}}{\sim} \lambda N \\ \epsilon_N &= \frac{1}{\sqrt{N}}.\end{aligned}$$

- Motivation for $\lambda_N = \left(1-e^{-\lambda}\right) N$: in a unit of time the vorticity of the initial N vortices reduces by a factor $e^{-\lambda}$, hence it must be replaced by $\left(1-e^{-\lambda}\right) N$ new vortices.
- Motivation for $\epsilon_N = \frac{1}{\sqrt{N}}$: the only way to get a truly stochastic solutions in the limit (turbulence should be described by stochastic solutions).

Scaling limit

The limit as $N \to \infty$ of the previous equation turn out to be

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= -\lambda \omega + \sqrt{\lambda} \cdot \textit{space-time white noise} \\ \omega|_{t=0} &= \text{space white noise} \end{aligned}$$

White noise in space is an invariant distribution. Compare with the known fact (Da Prato - Debussche; Albeverio - Ferrario) that white noise in space is invariant for

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \nabla^{\perp} \cdot \textit{solenoidal space-time white noise.}$$

Comments on the Ornstein-Uhlenbeck part

The reason for these facts is that white noise is invariant for the operator $u \cdot \nabla \omega$ and for both the Ornstein-Uhlenbeck equations

$$\partial_t \omega = -\lambda \omega + \sqrt{\lambda} \cdot \textit{space-time white noise}$$

$$\partial_t \omega = \Delta \omega + \nabla^{\perp} \cdot \textit{solenoidal space-time white noise}.$$

Small parameter limit

Summarizing, we started with a reasonable model of point vortex generation and depletion by friction; a suitable limit, scaled in such a way that stochasticity survives, leads to

$$\begin{aligned}\partial_t \omega + u \cdot \nabla \omega &= -\lambda \omega + \sqrt{\lambda} \cdot \textit{space-time white noise} \\ \omega|_{t=0} &= \text{space white noise}\end{aligned}$$

The limit $\lambda \to 0$ is a further simplification, which leads to the model

$$\begin{aligned}\partial_t \omega + u \cdot \nabla \omega &= 0 \\ \omega|_{t=0} &= \text{space white noise}\end{aligned}$$

which is considered in the above mentioned papers.

Does it have features of turbulence?

Construction apart, does the final result

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 \\ \omega|_{t=0} &= \text{space white noise} \end{aligned}$$

displays features of stationary inverse cascade?

Some yes (in a too perfect, idealized way). Others not (or not clear).

[First one: *stationarity*.]

Stationary inverse cascade 2D turbulence

Sample sentences/titles/keywords from the physics literature:

- <<Inverse cascade is nonintermittent, statistics of the velocity increments is close to Gaussian>> (P. Tabeling, Physics Reports 2002)
- <<The existence and robustness of the inverse energy cascade with its Gaussian, nonintermittent statistics and solid $k^{-5/3}$ scaling are well established.>> <<skewness$\sim$0,05>> (G. Boffetta and R. E. Ecke, Ann. Rev. Fluid Mech. 2012)
- Conformal invariance in hydrodynamic turbulence, G. Falkovich, Russian Math. Surveys 2007.
- coherent structures, condensate states (e.g. same ref.)

Gaussianity

At every time t, the solution constructed above is Gaussian.
This is an idealization, but not far from reality:

- <<Inverse cascade is nonintermittent, statistics of the velocity increments is close to Gaussian>> (P. Tabeling, Physics Reports 2002)
- <<The existence and robustness of the inverse energy cascade with its Gaussian, nonintermittent statistics and solid $k^{-5/3}$ scaling are well established.>> <<skewness$\sim$0,05>> (G. Boffetta and R. E. Ecke, Ann. Rev. Fluid Mech. 2012)

Conformal invariance

Recall that

$$\omega(t,\cdot) \in H^{-1-} \qquad \text{is space white noise.}$$

It turns out that, at every time t, the velocity field

$$u(t,\cdot) \in H^{0-}$$

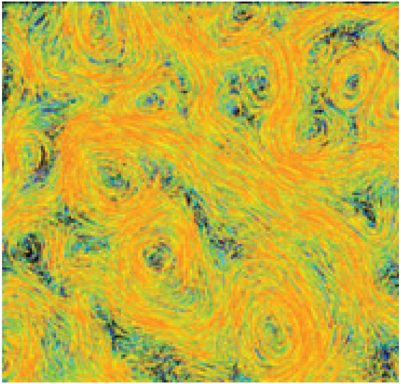
is a vector valued, solenoidal analog of the *Gaussian Free Field*. Having covariance given by $\int \nabla u \cdot \nabla v$, one can prove that its law is conformally invariant.

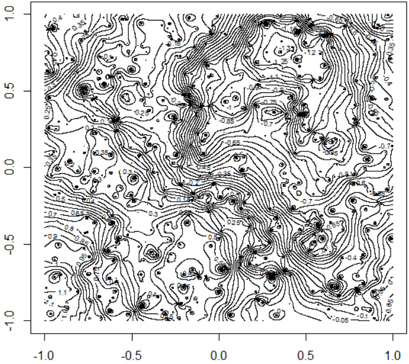
This *full* conformal invariance is again an idealization of the *traces* reported by:

- Conformal invariance in hydrodynamic turbulence, G. Falkovich, Russian Math. Surveys 2007.

Coherent structures, condensate states

There is a mild visual evidence. See samples of velocity field from experiment of Rivera-Ecke '05 versus a realization from point vortex approximation (2000 vortices) of white noise vorticity





Coherent structures, condensate states

The solenoidal Gaussian Free Field

$$u\left(t.\cdot\right) \in H^{0-}$$

seem to have concentration points ("thick points" in the language of the classical GFF). This however is just a static property of Gaussian fluctuations/concentration (no dynamics). Let us describe the claim.

Concentration points of the solenoidal Gaussian Free Field

(Joint work with Clara Antonucci and Francesco Grotto)
The following circulation field is well defined (as a Wiener integral)

$$\Gamma(x,r) = \frac{1}{2\pi r}\int_{\partial B(x,r)} u \cdot ds.$$

It has an Hölder continuous version, for $(x,r) \in \mathbb{T}^2 \times (0,1)$. Given a continuous sample $(x,r) \mapsto \Gamma(x,r)$, for a.e. $x \in \mathbb{T}^2$ we have

$$\lim_{r\to 0} \frac{\Gamma(x,r)}{\sqrt{\log \frac{1}{r}}} = 0.$$

But it seems that, for every $a \in (0,2)$,

$$\left\{ x \in \mathbb{T}^2 : \lim_{r\to 0} \frac{\Gamma(x,r)}{\sqrt{\log \frac{1}{r}}} = a \right\}$$

in non empty and has a certain Hausdorff dimension $h(a)$ (we have full proof of the upper bound and great part of the proof of the lower bound).

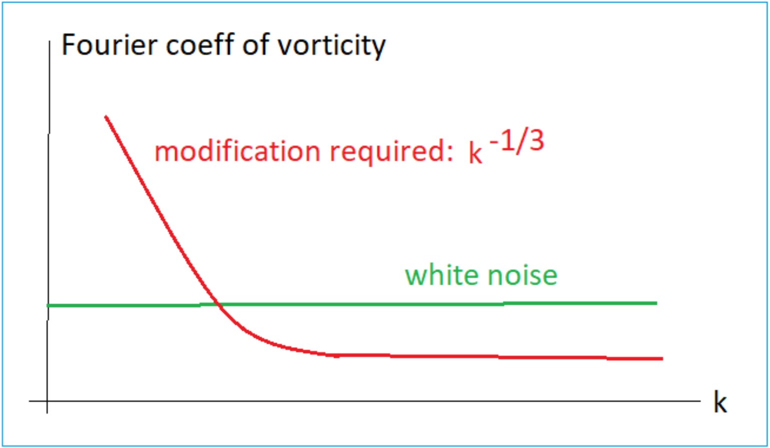
Inverse cascade, power spectrum?

- The model above was constructed simulating inverse cascade by means of point vortices. However, at present I have not found any evidence of inverse cascade in the final model.
- Real and numerical experiments show a power law energy spectrum

$$E(k) \sim k^{-5/3} \qquad \text{for} \qquad k_{\text{frict}} \leq k \leq k_{\text{inject}}.$$

The above model *does not* have this property (up to fluctuations, we have $E(k) \sim k^{-3/3}$).

Steepening of Fourier amplitudes



Internal noise

- This is a general question, in principle not necessarily related to turbulence.
- The question is: which is the meaning of noise in fluid mechanic models?
- Most sentences in the literature mention "external noise". My best interpretation of "external noise" is "hidden variables".
- But, especially related to turbulent regimes, there is the idea of "internal noise".

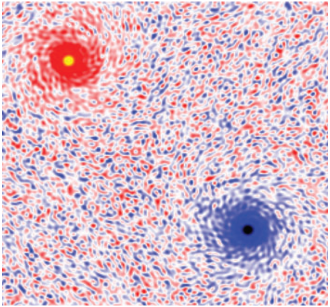
It simply means separation of scales; the smallest ones acting on the largest one as a noise.
[Another interpretation maybe is thermal noise, but I do not think so.]

Internal noise

Perturb the previous model $\partial_t \omega + u \cdot \nabla \omega = 0$ by some large scale structures:

$$\omega = \omega_{\text{large}} + \omega_{\text{white}}$$

(a background white noise with some large scale structure inside)



Example of perturbation

If we want to maintain white noise as a reference measure (for mathematical convenience) we may think that we consider the initial condition $\omega|_{t=0}$ distributed as

$$\rho_0(\omega)\,\mu_{\text{white}}(d\omega)$$

where $\mu_{\text{white}}(d\omega)$ is the law of white noise ω_{white}. Take for instance

$$\rho_0(\omega) \sim \frac{1}{Z}\exp\left(-\text{dist}\left(\omega, \omega_{\text{large}}|_{t=0}\right)\right).$$

Internal noise

Assume $\omega_{\text{large}}(t)$ satisfies

$$\partial_t \omega_{\text{large}} + (u_{\text{large}} + u_{\text{white}}) \cdot \nabla \omega_{\text{large}} = 0$$

and ω_{white} satisfies

$$\partial_t \omega_{\text{white}} + (u_{\text{large}} + u_{\text{white}}) \cdot \nabla \omega_{\text{white}} = 0.$$

Then

$$\partial_t (\omega_{\text{large}} + \omega_{\text{white}}) + (u_{\text{large}} + u_{\text{white}}) \cdot \nabla (\omega_{\text{large}} + \omega_{\text{white}}) = 0.$$

We have a decomposition of vorticity dynamics (this is the strategy to investigate interacting vortex patches, for instance).

Internal noise

The first claim, only heuristic at present, is that the solution of

$$\partial_t \omega_{\text{white}} + \left(u_{\text{large}} + u_{\text{white}}\right) \cdot \nabla \omega_{\text{white}} = 0$$

suitably rescaled (accelerated) in time, behaves as space-time white noise (ω_{white} a priori is only space-white noise, at every given time).

Internal noise

If ω_{large} is a slow variable compared to ω_{white}, the time rescaling of ω_{white} could not destroy the structure of the equation for ω_{large}, that was

$$\partial_t \omega_{\text{large}} + (u_{\text{large}} + u_{\text{white}}) \cdot \nabla \omega_{\text{large}} = 0$$

and lead to the stochastic equation

$$\partial_t \omega_{\text{large}} + u_{\text{large}} \cdot \nabla \omega_{\text{large}} + \sum_{k \in \mathbb{Z}^2} \sigma_k (x) \cdot \nabla \omega_{\text{large}} \circ \beta_k' (t) = 0$$

where we have denoted by $\sum_{k \in \mathbb{Z}^2} \sigma_k (x) \beta_k' (t)$ the rescaled process u_{white}, white noise in time, solenoidal Gaussian Free Field in space.

Internal noise

The previous derivation is only heuristic (opposite to the first main part of the talk). In
F. Flandoli, D. Luo, ρ-white noise solution to 2D stochastic Euler equations, arxiv
we have investigated the equation

$$\partial_t \omega + u \cdot \nabla \omega + \sum_{k \in \mathbb{Z}^2} \sigma_k(x) \cdot \nabla \omega \circ \beta_k'(t) = 0$$

when

$$|\sigma_k| \sim \frac{1}{|k|^{1+\epsilon}}, \qquad \epsilon > 0$$

(see also Brzezniak-F-Maurelli, Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity, ARMA 2016).
Solenoidal Gaussian Free Field is the limit case $\epsilon = 0$; it will be investigated in a work in preparation with Dejun Luo, using a renormalization argument.

A remark on regularization by noise

It is known that for linear transport equations with rough coefficients (F-Gubinelli-Priola, 2010), vector-valued similar linear models (F-Maurelli-Neklyudov 2014), point vortices (F-Gubinelli-Priola, 2011) and Vlasov-Poisson point charges (Delarue-F-Vincenzi 2014) a noise of the form

$$\sum_{k\in\mathbb{Z}^2} \sigma_k(x) \cdot \nabla\omega \circ \beta_k'(t)$$

has a regularization effect. For 2D Euler equation we still do not know. The internal noise interpretation above corresponds to the idea that singularities in fluids, if any, happen in the transient regime, not in stationary turbulence (see also F-Romito 2002 on Caffarelli-Kohn-Nirenberg theory under noise).
Internal noise due to turbulence should have a regularizing effect.

Summary

- Rigorous results on 2D Euler equations with white noise initial conditions have been proved.
- The purpose of the talk has been to discuss stationary inverse cascade showing that 2D Euler equations with white noise initial conditions may be related.
- Moreover, the final model satisfies properties claimed in the literature on turbulence.
- The same model may be the basis for the investigation of internal noise.
- Open: understanding inverse cascade, spectrum, rigorous approach to internal noise.

Thank you for your attention