Pathwise well-posedness of stochastic porous media equations with nonlinear, conservative noise

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I. The equation

We will discuss a well-posedness theory for porous media and fast diffusion equations with nonlinear, conservative noise:

$$\partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ \mathrm{d}z_t) \text{ in } \mathbb{T}^d \times (0,\infty),$$

for a diffusion exponent $m \in (0,\infty)$ on the d-dimensional torus.

- The noise is a n-dimensional, α-Hölder continuous, and geometric rough path z.
- The equations are not a priori well-posed due to the rough driving signal. The nonlinearity precludes transformation methods.
- The methods are motivated by the theory of stochastic viscosity solutions of Lions and Souganidis [19, 20, 21, 22, 23], and rely on the rough path analysis of Lyons [24].

Hydrodynamic limit of the zero range process

The zero range process is an interacting particle process on $\mathbb{Z}^d/n\mathbb{Z}^d$. It is "zero range" in the sense that the rate of diffusion from a given point is determined solely by the number of particles occupying that point.

It was shown by Ferrari, Presutti, and Vares [7] that the hydrodynamic limit of a zero range particle process satisfies a nonlinear diffusion equation of the type

$$\partial_t u = \Delta\Phi(u) \text{ in } \mathbb{T}^d \times (0,\infty),$$

where Φ is the mean local jump rate.

Fluctuating hydrodynamics of the zero range process

The fluctuating hydrodynamics of the zero range process about its hydrodynamic limit were subsequently studied by Ferrari, Presutti, and Vares [8], and were informally shown by Dirr, Stamatakis, and Zimmer [3] to satisfy a stochastic nonlinear diffusion equation of the type

$$\partial_t u = \Delta\left(\Phi(u)\right) + \nabla \cdot \left(\sqrt{\Phi(u)}\mathcal{N}\right) \text{ in } \mathbb{T}^d \times (0,\infty),$$

where $\mathcal{N}$ is a space-time white noise and σ is a bulk diffusion coefficient.

Mean field games (I)

Consider an L-dimensional system of mean field stochastic differential equations, for $i \in \{0, \ldots, L\}$,

$$\mathrm{d}X_t^i = A^L(X_t^i, \frac{1}{L}\sum_{j\neq i}\delta_{X_t^j}) \circ \mathrm{d}B_t + \Sigma^L(\frac{1}{L}\sum_{j\neq i}\delta_{X_t^j})\,\mathrm{d}W_t^i,$$

for $L \geq 1$ and for $\{B_t^i\}_{i=1}^n$ and $\{W_t^i\}_{i=1}^d$ independent Brownian motions.

The coefficients $\{A^L\}_{L\geq 1}$ and $\{\Sigma^L\}_{L\geq 1}$ are defined and continuous on the space $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ and $\mathcal{P}(\mathbb{T}^d)$, for $\mathcal{P}(\mathbb{T}^d)$ the space of probability measures with the topology of weak convergence.

Mean field games (II)

The theory of mean field games, as introduced by Lasry and Lions [14, 15, 16], informally shows that that the density m of the empirical law of the solution $X_t = (X_t^1, \ldots, X_t^L)$, in the mean field limit $L \to \infty$, evolves according to an equation of the form

$$\partial_t m = \frac{1}{2}\Delta\left(\sigma^2(m)m\right) + \nabla \cdot (A(x,m)m \circ \mathrm{d}B_t),$$

provided $\{A^L\}_{\{L\geq 1\}}$ and $\{\Sigma^L\}_{\{L\geq 1\}}$ converge, as $L \to \infty$, to local functions A and σ.

Dean-Kawasaki model

The Dean-Kawasaki model is an approximation for the diffusion of particles subject to thermal advection in a fluctuating fluid. The model was proposed by Dean [2], Kawasaki [13] and Marconi and Tarazona [25], and recently studied by Donev, Fai, and Vanden-Eijnden [4].

The density of the particles c evolves according to the stochastic equation

$$\partial_t c = \sigma \Delta c + \nabla \cdot \left(cv + \sqrt{2\sigma c}\mathcal{N} \right),$$

where $\sigma > 0$ is a diffusion coefficient, v is a smooth and divergence free velocity field, and $\mathcal{N}$ is a space-time white noise.

The thin film equation

Grün, Mecke, and Rauscher [12] have proposed a stochastic model for the evolution of a thin film consisting of an incompressible Newtonian liquid on a flat d-dimensional substrate.

The thickness h of the substrate satisfies

$$\partial_t h = \nabla \cdot \left(h^n \nabla \left(\frac{1}{3} \Phi'(h) - \gamma \Delta h \right) \right) + \nabla \cdot \left(\frac{h^{\frac{3}{2}}}{3} \mathcal{N} \right),$$

where Φ is the effective interface potential, $\gamma > 0$ is the surface tension coefficient, $\mathcal{N}$ is a space-time white noise, and $n > 0$ describes the mobility function.

The entropy formulation

Consider a deterministic scalar conservation law of the form

$$\partial_t u = \Delta u^{[m]} + \nabla \cdot f(x,u).$$

The entropy formulation of this equation is based upon studying the ensemble equations satisfies by compositions $S(u)$, for smooth and convex entropies S satisfying $S(0) = S'(0) = 0$:

$$\begin{aligned}\partial_t S(u) =& \nabla \cdot \left(m \left| u \right|^{m-1} \nabla S(u) \right) - S''(u) \frac{4m}{(m+1)^2} \left| \nabla u^{\left[\frac{m+1}{2}\right]} \right|^2 \\ & + \partial_\xi f(x,u) \nabla S(u) + \nabla_x f(x,u) S'(u).\end{aligned}$$

The kinetic function

In the kinetic formulation, the ensemble of equations defined by the collection of entropies $\{S\}$ is replaced by a single equation in $(d+1)$-variables.

Define the kinetic function $\overline{\chi}:\mathbb{R}^2\to\mathbb{R}$ by

$$\overline{\chi}(s,\xi) := \begin{cases} 1 & \text{if } 0<\xi<s, \\ -1 & \text{if } s<\xi<0, \\ 0 & \text{else.} \end{cases}$$

The kinetic function χ of an arbitrary scalar function u is then

$$\chi(x,\xi,t) := \overline{\chi}(u(x,t),\xi) = \begin{cases} 1 & \text{if } 0<\xi<u(x,t), \\ -1 & \text{if } u(x,t)<\xi<0, \\ 0 & \text{else.} \end{cases}$$

The kinetic formulation

For the d-dimensional torus $\mathbb{T}^d$, for the solution u of

$$\partial_t u = \Delta u^{[m]} + \nabla \cdot f(x,u) \text{ in } \mathbb{T}^d \times (0,\infty),$$

an informal computation proves that the kinetic function χ of u satisfies

$$\begin{aligned}\partial_t \chi =& m\left|\xi\right|^{m-1}\Delta_x \chi + \partial_\xi q \\ &+ \partial_\xi f(x,\xi)\nabla_x \chi - \nabla_x f(x,\xi)\partial_\xi \chi,\end{aligned}$$

for the parabolic defect measure

$$q(x,\xi,t) := \delta_0(u(x,t)-\xi)\frac{4m}{(m+1)^2}\left|\nabla u^{\left[\frac{m+1}{2}\right]}\right|^2.$$

The derivation of the kinetic formulation

The kinetic formulation was introduced by Perthame [26] and Chen and Perthame [1].

The kinetic formulation:

- Strictly generalizes the notion of an entropy solution.
- Characterizes solutions with initial data in L^1.
- Is informally derived by multiplying the equation with $S'(\xi)$, integrating in ξ, and using the distributional equalities

$$\nabla_x \chi = \delta_0(u-\xi)\nabla u \ \text{ and } \ \partial_\xi \chi = \delta_0(\xi) - \delta_0(u-\xi).$$

The kinetic formulation of the stochastic equation

For a smooth path z, the kinetic formulation is

$$\begin{aligned}\partial_t \chi =& m\,|\xi|^{m-1}\,\Delta_x \chi + \partial_\xi q \\ &+ \partial_\xi A(x,\xi)\dot{z}_t \cdot \nabla_x \chi - \left(\nabla_x \cdot (A(x,\xi)\cdot \dot{z}_t)\right)\partial_\xi \chi,\end{aligned}$$

for the parabolic defect measure

$$q(x,\xi,t) := \delta_0(u(x,t)-\xi)\frac{4m}{(m+1)^2}\left|\nabla u^{\left[\frac{m+1}{2}\right]}(x,t)\right|^2.$$

- Regularity is encoded by the measure.
- Nonlinear due to the nonlinearity of the kinetic function and the nonlinear dependence of the measure.

Transport by noise

An essential feature of the corresponding kinetic formulation

$$\begin{aligned}\partial_t\chi =& m\,|\xi|^{m-1}\,\Delta_x\chi + \partial_\xi q \\ &+ \partial_\xi A(x,\xi)\dot{z}_t\cdot\nabla_x\chi - \left(\nabla_x\cdot\left(A(x,\xi)\cdot\dot{z}_t\right)\right)\partial_\xi\chi,\end{aligned}$$

is that the noise enters as a linear transport.

Furthermore:

- The transport is well-defined for rough paths z.
- Due to the conservative structure, the corresponding stochastic characteristics preserve the underlying Lebesgue measure.

The stochastic characteristics

The associated system of stochastic characteristics are understood as a system of rough differential equations:

$$\begin{cases} \mathrm{d}X^{x,\xi}_{t_0,t} = -\partial_\xi A(X^{x,\xi}_{t_0,t}, \Xi^{x,\xi}_{t_0,t}) \circ \mathrm{d}z_t & \text{in } (t_0,\infty), \\ \mathrm{d}\Xi^{x,\xi}_{t_0,t} = \nabla_x \cdot A(X^{x,\xi}_{t_0,t}, \Xi^{x,\xi}_{t_0,t}) \circ \mathrm{d}z_t & \text{in } (t_0,\infty), \\ (X^{x,\xi}_{t_0,t_0}, \Xi^{x,\xi}_{t_0,t_0}) = (x,\xi). \end{cases}$$

- Stability of solutions up to their second derivatives requires that $\partial_\xi A$ and $(\nabla_x \cdot A)$ are in $\mathrm{C}^{(\frac{1}{\alpha}+2)+}$.
- Therefore, we essentially require $A(x,\xi) \in \mathrm{C}^{(\frac{1}{\alpha}+3)+}$ and allow for linear growth in the ξ-variable.

The transported kinetic function

The transport of the kinetic function χ by the characteristics $(X^{x,\xi}_{t_0,t}, \Xi^{x,\xi}_{t_0,t})$ beginning from $t_0 \geq 0$ and $(x,\xi) \in \mathbb{T}^d \times \mathbb{R}$, is defined by

$$\tilde{\chi}_{t_0,t}(x,\xi) := \chi(X^{x,\xi}_{t_0,t}, \Xi^{x,\xi}_{t_0,t}, t).$$

Informally, this function satisfies

$$\partial_t \tilde{\chi}_{t_0,t} = m \left| \Xi^{x,\xi}_{t_0,t} \right|^{m-1} \tilde{\Delta}_x \tilde{\chi} + \tilde{\partial}_\xi \tilde{q}_{t_0,t},$$

where $\tilde{q}$ is the translated parabolic defect measure

$$\tilde{q}(x,\xi)_{t_0,t} := q(X^{x,\xi}_{t_0,t}, \Xi^{x,\xi}_{t_0,t}, t),$$

and where $\tilde{\Delta}_x$ and $\tilde{\partial}_\xi$ are not full derivatives.

The weak formulation

For the d-dimensinal unit torus $\mathbb{T}^d$, the corresponding weak formulation of the equation is then, for each $\rho \in \mathrm{C}^\infty(\mathbb{T}^d \times \mathbb{R})$, for each $s < t$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi \cdot \rho(Y^{x,\xi}_{s,r-s}, \Pi^{x,\xi}_{s,r-s}) \,\mathrm{d}x \,\mathrm{d}\xi \Big|_{r=s}^{r=t} \\
&= \int_s^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} m \,|\xi|^{m-1} \chi \cdot \Delta_x \rho(Y^{x,\xi}_{s,r-s}, \Pi^{x,\xi}_{s,r-s}) \,\mathrm{d}x \,\mathrm{d}\xi \,\mathrm{d}r \\
&\quad - \int_s^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} q \cdot \partial_\xi \rho(Y^{x,\xi}_{s,r-s}, \Pi^{x,\xi}_{s,r-s}) \,\mathrm{d}x \,\mathrm{d}\xi \,\mathrm{d}r,
\end{aligned}$$

where $(Y^{x,\xi}_{s,r-s}, \Pi^{x,\xi}_{s,r-s})$ denote the characteristics inverse to $(X^{x,\xi}_{s,r}, \Xi^{x,\xi}_{s,r})$.

- The derivation relies crucially on the conservative property of the characteristics.

Pathwise kinetic solutions (I)

Definition

For $u_0 \in L^2(\mathbb{T}^d)$, a *pathwise kinetic solution* is a function

$$u \in L^\infty([0,\infty); L^2(\mathbb{T}^d)),$$

that satisfies the following three properties:
(i) For each $T > 0$,

$$u^{\left[\frac{m+1}{2}\right]} \in L^2([0,T]; H^1(\mathbb{T}^d)).$$

In particular, for each $T > 0$, the parabolic defect measure

$$q(x,\xi,t) := \frac{4m}{(m+1)^2}\delta_0(\xi - u(x,t))\left|\nabla u^{\left[\frac{m+1}{2}\right]}\right|^2,$$

is finite on $\mathbb{T}^d \times \mathbb{R} \times (0,T)$.

Pathwise kinetic solutions (II)

Definition

(ii) There exists a nonnegative entropy defect measure p on $\mathbb{T}^d \times \mathbb{R} \times (0,\infty)$, which is finite on $\mathbb{T}^d \times \mathbb{R} \times (0,T)$, for each $T>0$, such that, for a set of Lebesgue measure zero $\mathcal{N} \subset (0,\infty)$, for every $s \leq t \in [0,\infty) \setminus \mathcal{N}$, the kinetic function χ of u satisfies, for every $\rho \in \mathrm{C}_c^\infty(\mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi(x,\xi,r) \cdot \rho(Y^{x,\xi}_{r,r-s}, \Pi^{x,\xi}_{r,r-s}) \,\mathrm{d}x\,\mathrm{d}\xi \Big|_{r=s}^{r=t} \\
&= \int_s^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} m\,|\xi|^{m-1}\,\chi \cdot \Delta\rho(Y^{x,\xi}_{r,r-s}, \Pi^{x,\xi}_{r,r-s}) \,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}r \\
&\quad - \int_s^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p+q) \cdot \partial_\xi \rho(Y^{x,\xi}_{r,r-s}, \Pi^{x,\xi}_{r,r-s}) \,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}r.
\end{aligned}$$

The initial condition is imposed at $s=0$.

Pathwise kinetic solutions (III)

Definition

(iii) The following integration by parts formula is satisfied. For each $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times \mathbb{R} \times [0,\infty))$, for each $t \in [0,\infty) \setminus \mathcal{N}$,

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} \frac{m+1}{2} \left|\xi\right|^{\frac{m-1}{2}} \chi(x,\xi,r) \nabla\psi(x,\xi,r) \,\mathrm{d}x \,\mathrm{d}\xi \,\mathrm{d}r$$
$$= - \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} \nabla u^{\left[\frac{m+1}{2}\right]} \psi(x,u(x,r),r) \,\mathrm{d}x \,\mathrm{d}r.$$

- The degeneracy of the diffusion makes it necessary to postulate the integration by parts formula.
- In particular, it is unnecessary if $m = 1$.

Existence of pathwise kinetic solutions

Theorem [6] (F., Gess)

Let $m \in (0, \infty)$. Suppose the z is an α-Hölder geometric rough path, for $\alpha \in (0, \frac{1}{2})$, and, for $\gamma > \frac{1}{\alpha}$,

$$\nabla_x \cdot A(x, \xi) \in \mathrm{C}_b^{\gamma+2} \text{ and } \partial_\xi A(x, \xi) \in \mathrm{C}_b^{\gamma+2}.$$

For every $u_0 \in L^2(\mathbb{T}^d)$, there exists a pathwise kinetic solution of

$$\begin{cases} \partial_t u = \Delta u^{[m]} + \nabla \cdot (A(x, u) \circ \mathrm{d}z_t) & \text{in } \mathbb{T}^d \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{T}^d \times \{0\}. \end{cases}$$

- The integrability is not optimal, and can be extended to $L^1(\mathbb{T}^d)$, modifying the definition of solution.
- The proof is based on a new estimate for the kinetic function in $W^{s,1}$, for $s \in \left(0, \frac{2}{m+1} \wedge 1\right)$.

Regularity of the kinetic function

Proposition [6] (F., Gess)

Let $u_0 \in L^2(\mathbb{T}^d)$, and let u be a pathwise kinetic solution with initial data u_0 and kinetic function χ. For each $s \in (0, \frac{2}{m+1} \wedge 1)$ and $T \geq 0$, there exists $C = C(m, d, T, s) > 0$ such that

$$\|\chi\|_{L^1\left([0,T];W^{s,1}(\mathbb{T}^d\times\mathbb{R})\right)} \leq C\left(1 + \|u_0\|_{L^1(\mathbb{T}^d)} + \|u_0\|_{L^1(\mathbb{T}^d)}^{(m+1)\vee 2} + \|u_0\|_{L^2(\mathbb{T}^d)}^2\right).$$

- The proof combines a uniform BV-estimate in the velocity variable with the spatial regularity implied by the parabolic defect measure.
- Some elements of the spatial regularity are based on the work of Ebmeyer [5].

Uniqueness of pathwise kinetic solutions

Theorem [6] (F., Gess)

Let $m \in (0,\infty)$. Suppose the z is an α-Hölder, geometric rough path, for $\alpha \in (0,\frac{1}{2})$, and that, for $\gamma > \frac{1}{\alpha}$,

$$\nabla_x A(x,\xi) \in \mathrm{C}_b^{\gamma+2} \quad \text{and} \quad \partial_\xi A(x,\xi) \in \mathrm{C}_b^{\gamma+2}.$$

Let $u_0^1, u_0^2 \in L_+^2(\mathbb{T}^d)$. If u^1 and u^2 are pathwise kinetic solutions with initial data u_0^1 and u_0^2, then

$$\left\|u^1 - u^2\right\|_{L^\infty([0,\infty);L^1(\mathbb{T}^d))} \leq \left\|u_0^1 - u_0^2\right\|_{L^1(\mathbb{T}^d)}.$$

In particular, pathwise kinetic solutions are unique.

- This work extends results of Lions, Perthame and Souganidis [17, 18] and the Gess and Souganidis [9, 10, 11], who worked in analogous x-independent and first-order settings.

The proof of uniqueness

Some aspects of the proof:

- The characteristics link the spatial and velocity variables.
- It is necessary to use rough path estimates and a time-splitting argument to handle errors arising from the transport by the stochastic characteristics.
- It is necessary to use an optimal estimate for a singular moment of the entropy and parabolic defect measures for small diffusion exponents $m \in (0,1) \cup (1,2]$, which relies upon the nonnegativity of the initial data.
- The integrability is not optimal.

Singular moments of the defect measures

Proposition [6] (F., Gess)

Let $u_0 \in L^2_+(\mathbb{T}^d)$. Suppose that u is a pathwise kinetic solution with initial data u_0 and entropy and parabolic defect measures p and q. For each $T > 0$, there exists $C = C(m, d, T) > 0$ such that

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\xi|^{-1} (p+q) \, \mathrm{d}x \, \mathrm{d}\xi \, \mathrm{d}r \leq C \left(1 + \|u_0\|^2_{L^2(\mathbb{T}^d)}\right).$$

- Informally, this implies regularity for $u^{[\frac{m}{2}]}$.
- The estimate is false, in general, for signed initial data. The failure is deterministic, and can be seen by considering the heat equation with initial data that is locally linear near the origin.

The existence of a random dynamical system

Theorem [6] (F., Gess)

Suppose that the noise $t \in [0,\infty) \mapsto z_t(\omega)$ arises from the sample paths of a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on a probability space $\omega \in (\Omega, \mathcal{F}, \mathbb{P})$. Pathwise kinetic solutions generate a random dynamical system on $L^2_+(\mathbb{T}^d)$ in the sense that, for every $u_0 \in L^2_+(\mathbb{T}^d)$, for almost every $\omega \in \Omega$,

$$u(u_0, s, t; z_\cdot(\omega)) = u(u_0, 0, t-s; z_{\cdot+s}(\omega)) \text{ for every } 0 \leq s \leq t < \infty.$$

- The random dynamical system is an almost sure semigroup property for the equation.
- These are the first results proving the existence of a random dynamical system for a nonlinear SPDE with x-dependent, nonlinear noise.

Thanks

Thank you.

Bibliography I

G.-Q. Chen and B. Perthame.
Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations.
Ann. Inst. H. Poincaré Anal. Non Linéaire, 20(4):645–668, 2003.

D. S. Dean.
Langevin equation for the density of a system of interacting langevin processes.
Journal of Physics A: Mathematical and General, 29(24):L613, 1996.

N. Dirr, M. Stamatakis, and J. Zimmer.
Entropic and gradient flow formulations for nonlinear diffusion.
J. Math. Phys., 57(8):081505, 13, 2016.

A. Donev, T. G. Fai, and E. Vanden-Eijnden.
A reversible mesoscopic model of diffusion in liquids: from giant fluctuations to fick's law.
Journal of Statistical Mechanics: Theory and Experiment, P04004, 2014.

C. Ebmeyer.
Regularity in Sobolev spaces for the fast diffusion and the porous medium equation.
J. Math. Anal. Appl., 307(1):134–152, 2005.

B. Fehrman and B. Gess.
Well-posedness of stochastic porous media equations with nonlinear, conservative noise.
arXiv e-prints, May 2018.

P. A. Ferrari, E. Presutti, and M. E. Vares.
Local equilibrium for a one-dimensional zero range process.
Stochastic Process. Appl., 26(1):31–45, 1987.

Bibliography II

P. A. Ferrari, E. Presutti, and M. E. Vares.
Nonequilibrium fluctuations for a zero range process.
Ann. Inst. H. Poincaré Probab. Statist., 24(2):237–268, 1988.

B. Gess and P. E. Souganidis.
Scalar conservation laws with multiple rough fluxes.
Commun. Math. Sci., 13(6):1569–1597, 2015.

B. Gess and P.E. Souganidis.
Long-time behavior, invariant measures and regularizing effects for stochastic scalar-conservation laws.
arXiv:1411.5939, 2016.

E. Gess and P. E. Souganidis.
Stochastic non-isotropic degenerate parabolic-hyperbolic equations.
arXiv:1611.01303, 2016.

G. Grün, K. Mecke, and M. Rauscher.
Thin-film flow influenced by thermal noise.
Journal of Statistical Physics, 122(6):1261–1291, Mar 2006.

K. Kawasaki.
Microscopic analyses of the dynamical density functional equation of dense fluids.
J. Statist. Phys., 93(3-4):527–546, 1998.

J.-M. Lasry and P.-L. Lions.
Jeux à champ moyen. I. Le cas stationnaire.
C. R. Math. Acad. Sci. Paris, 343(9):619–625, 2006.

Bibliography III

J.-M. Lasry and P.-L. Lions.
Jeux à champ moyen. II. Horizon fini et contrôle optimal.
C. R. Math. Acad. Sci. Paris, 343(10):679–684, 2006.

J.-M. Lasry and P.-L. Lions.
Mean field games.
Jpn. J. Math., 2(1):229–260, 2007.

P.-L. Lions, B. Perthame, and P. E. Souganidis.
Scalar conservation laws with rough (stochastic) fluxes.
Stoch. Partial Differ. Equ. Anal. Comput., 1(4):664–686, 2013.

P.-L. Lions, B. Perthame, and P. E. Souganidis.
Scalar conservation laws with rough (stochastic) fluxes: the spatially dependent case.
Stoch. Partial Differ. Equ. Anal. Comput., 2(4):517–538, 2014.

P.-L. Lions and P. E. Souganidis.
Fully nonlinear stochastic partial differential equations.
C. R. Acad. Sci. Paris Sér. I Math., 326(9):1085–1092, 1998.

P.-L. Lions and P. E. Souganidis.
Fully nonlinear stochastic partial differential equations: non-smooth equations and applications.
C. R. Acad. Sci. Paris Sér. I Math., 327(8):735–741, 1998.

P.-L. Lions and P. E. Souganidis.
Fully nonlinear stochastic pde with semilinear stochastic dependence.
C. R. Acad. Sci. Paris Sér. I Math., 331(8):617–624, 2000.

Bibliography IV

P.-L. Lions and P. E. Souganidis.
Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations.
C. R. Acad. Sci. Paris Sér. I Math., 331(10):783–790, 2000.

P.-L. Lions and P. E. Souganidis.
Viscosity solutions of fully nonlinear stochastic partial differential equations.
Sūrikaisekikenkyūsho Kōkyūroku, (1287):58–65, 2002.
Viscosity solutions of differential equations and related topics (Japanese) (Kyoto, 2001).

Terry J. Lyons.
Differential equations driven by rough signals.
Rev. Mat. Iberoamericana, 14(2):215–310, 1998.

U. M. B. Marconi and P. Tarazona.
Dynamic density functional theory of fluids.
The Journal of Chemical Physics, 110(16):8032-8044, 1999.

B. Perthame.
Kinetic formulation of conservation laws, volume 21 of *Oxford Lecture Series in Mathematics and its Applications*.
Oxford University Press, Oxford, 2002.