Integration by parts formulae for the laws of Bessel bridges, and stochastic PDEs with reflection

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(in joint work with Lorenzo Zambotti)

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For all $\delta > 0$ and $a \geq 0$, let $P^{\delta}_{a,a}$ be the law, on $L^2(0,1)$, of the δ dimensional Bessel bridge between a and a. We denote by $E^{\delta}_{a,a}$ the corresponding expectation operator.

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A first step: is to see how the laws $P^{\delta}_{a,a}$ behave under infinitesimal shift. More precisely, we compute integration by parts formulae for the laws of the Bessel bridges on a set of paths.

Integration by parts formulae (IbPF)

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For all $\Phi \in L^2(0,1) \to \mathbb{R}_+$ differentiable, and $h \in C_c^2(0,1)$, let:

$$\partial_h \Phi(X) := \langle \nabla \Phi(X), h \rangle = \lim_{\epsilon \to 0} \frac{\Phi(X + \epsilon h) - \Phi(X)}{\epsilon}, \quad X \in L^2(0,1).$$

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Question: Can we compute:

$$E^{\delta}_{a,a}[\partial_h \Phi(X)] \quad ?$$

A toy model

Recall the family of Borel measures on $\mathbb{R}_+$, $(\mu_\alpha)_{\alpha \geq 0}$,defined by:

$$\forall \alpha > 0, \quad \mu_\alpha(\,\mathrm{d}x) := \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}x,$$

and

$$\mu_0(dx) := \delta_0(\,\mathrm{d}x).$$

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Proposition (Integration by parts formulae)

For all $f \in C_c^1(\mathbb{R}_+)$, we have:

$$\forall \alpha > 1, \quad \int_{\mathbb{R}_+} f'(x)\,\mathrm{d}\mu_\alpha(x) = -(\alpha - 1)\int_{\mathbb{R}_+} \frac{f(x)}{x}\,\mathrm{d}\mu_\alpha(x),$$

and

$$\int_{\mathbb{R}_+} f'(x)\,\mathrm{d}\mu_1(x) = -f(0).$$

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Recall that, for $\alpha \in (-1, 0)$, we define μ_α as a Schwartz distribution on $\mathbb{R}_+$:

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For all $f \in C_c^\infty(\mathbb{R}_+)$, we have:

$$\forall \alpha \in (-1,0), \quad \mu_\alpha(f') = -(\alpha - 1) \, \mu_\alpha \left(\frac{f(x) - f(0) - x f'(0)}{x} \right).$$

IbPF for the Bessel bridges

For all $r \in (0,1)$, let $p_r^{a,\delta}(b)\,\mathrm{d}b$ be the law of X_r when $X \overset{(d)}{=} P^{\delta}_{a,a}$. Note that :

$$p_r^{a,\delta}(b) \underset{b \to 0}{\approx} b^{\delta-1}.$$

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Definition

For all $a, b \geq 0$, $\delta > 0$ and $r \in (0,1)$, let

$$\Sigma_{\delta,r}^{a}(\mathrm{d}X|b) := \frac{p_r^{a,\delta}(b)}{b^{\delta-1}} P_{a,a}^{\delta}\left[\mathrm{d}X | X_r = b\right].$$

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For all $b \geq 0$, $\Sigma^{a}_{\delta,r}(\,\mathrm{d}X|b)$ should be the *Revuz measure* of the local time, at level b, of the process $(u(t,r))_{t\geq 0}$, where u is the (speculative) solution of the SPDE with invariant measure $P^{\delta}_{a,a}$.

Theorem (Zambotti, 2000's)

Let $\Phi \in C_b^1\left(L^2(0,1)\right)$ *and* $h \in C_c^2(0,1)$*. Then for all* $a \geq 0$ *and* $\delta > 3$*:*

$$E_{a,a}^{\delta}[\partial_h \Phi(X)] = -E_{a,a}^{\delta}\left[\langle h'', X\rangle \Phi(X)\right] - 2\kappa(\delta) E_{a,a}^{\delta}[\langle h, X^{-3}\rangle \Phi(X)],$$

where $\kappa(\delta) = \frac{(\delta-1)(\delta-3)}{8} > 0$.

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$$E_{a,a}^{3}\left[\partial_h \Phi(X)\right] = -E_{a,a}^{3}\left[\langle h'', X\rangle \Phi(X)\right] - \frac{1}{2}\int_0^1 dr\, h(r)\, \Sigma_{3,r}^{a}(\Phi(X)|0)$$

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Guess: We still expect to have

$$E^{\delta}_{a,a}[\partial_h \Phi(X)] = -E^{\delta}_{a,a}\left[\langle h'', X\rangle \Phi(X)\right] + \text{a drift term.}$$

When $\delta < 3$, by conditioning :

$$E^{\delta}_{a,a}[\langle h, X^{-3}\rangle \Phi(X)] = \int_0^1 dr\ h(r) \int_0^{\infty} \mathrm{d}b\, b^{\delta-4}\, \Sigma^{a}_{\delta,r}(\Phi(X) |\, b\,)$$
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However, if $\delta < 3$, but $\delta > 2$, then:

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has a chance to converge... at least if Φ is "nice".

A space of functionals

Here is the main idea: in order to derive IbPFs for the laws $P^{\delta}_{a,a}$, we choose a class of functionals $\Phi : L^2(0,1) \to \mathbb{R}$ behaving nicely with these laws.

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Definition

Let $\mathcal{E}$ denote the vector space generated by functionals $\Phi : L^2(0,1) \to \mathbb{R}$ of the form

$$\Phi(X) := \exp\left(-\int_0^1 \theta_r X_r^2 \, \mathrm{d}r\right), \quad X \in L^2(0,1)$$

where $\theta : [0,1] \to \mathbb{R}_+$ is a Borel and bounded function.

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For $\Phi \in \mathcal{E}$, we can compute quantities such as $E^{\delta}_{a,a}[\partial_h \Phi(X)]$, or $\Sigma^{a}_{\delta,r}(\Phi(X)|b)$, semi-explicitly. In particular:

Remark

For all $r \in (0,1)$, $\Sigma^{a}_{\delta,r}(\Phi(X)|b)$ is a smooth function of b^2.

IbPF for δ-Bessel bridges, $\delta \in (1, 3)$

Theorem (EA, Zambotti)

Let $\delta \in (1, 3)$, $\Phi \in \mathcal{E}$ and $h \in C_c^2(0, 1)$. Then Φ satisfies an IbPF wrt $P^{\delta}_{a,a}$, with a drift term given by:

$$-2\kappa(\delta) \int_0^1 \mathrm{d}r \; h(r) \int_0^{\infty} \mathrm{d}b \, b^{\delta-4} \left(\Sigma^a_{\delta,r}(\Phi(X) | \, b \,) - \Sigma^a_{\delta,r}(\Phi(X) | \, 0 \,) \right).$$

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Remark

There is surprisingly no transition at $\delta = 2$. This is related to the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}b} \Sigma^a_{\delta,r}(\Phi(X) | \, b \,) \Big|_{b=0} = 0.$$

$\delta = 1$, the case of reflected Brownian bridges

Theorem (EA, Zambotti)

Let $\Phi \in \mathcal{E}$ and $h \in C_c^2(0,1)$. Then Φ satisfies an IbPF wrt $P^1_{a,a}$, with a drift term given by:

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h(r)\, \frac{\mathrm{d}^2}{\mathrm{d}b^2}\Sigma^a_{1,r}(\Phi(X)|b)\bigg|_{b=0}.$$

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Remark

For $\delta = 1$, similarly as for $\delta = 3$, the drift is local around $b = 0$

IbPF for δ-Bessel bridges, $\delta \in (0,1)$

Theorem (EA, Zambotti)

Let $\delta \in (0,1)$, $\Phi \in \mathcal{E}$ *and* $h \in C_c^2(0,1)$. *Then* Φ *satisfies an IbPF wrt* $P_{a,a}^{\delta}$, *with a drift term given by:*

$$- 2\,\kappa(\delta) \int_0^1 \mathrm{d}r \; h(r) \int_0^\infty \mathrm{d}b \, b^{\delta-4}$$

$$\left(\Sigma^a_{\delta,r}(\Phi(X) |\, b\,) - \Sigma^a_{\delta,r}(\Phi(X) |\, 0\,) - \frac{b^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Sigma^a_{\delta,r}(\Phi(X) |\, x\,)\Big|_{x=0} \right).$$

Conjecture for the form of the corresponding SPDEs

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More precisely, for all $x \in (0,1)$, there should exist a process $(\ell^b_t(x))_{b \geq 0, t \geq 0}$, such that:

$$\forall f : \mathbb{R}_+ \to \mathbb{R}_+, \quad \int_0^t f(u(s,x))\, ds = \int_0^{+\infty} f(b)\, \ell^b_t(x)\, b^{\delta-1}\, \mathrm{d}b.$$

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Then, for $\delta \in (1,3)$, the SPDE would be formally given by:

$$\partial_t u = \frac{1}{2}\partial^2_x u + \kappa(\delta) \int_0^{+\infty} \mathrm{d}b\, b^{\delta-4} \left(\mathrm{d}\ell^b_t(x) - \mathrm{d}\ell^0_t(x) \right) + \xi$$

Similarly, for $\delta \in (0,1)$, the drift in the SPDE would be formally given by:

$$\kappa(\delta) \int_0^{+\infty} \mathrm{d}b \, b^{\delta-4} \left(\mathrm{d}\ell_t^b(x) - \mathrm{d}\ell_t^0(x) - \frac{b^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}a^2} \, \mathrm{d}\ell_t^a(x) \Big|_{a=0} \right).$$

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Finally, for $\delta = 1$, the SPDE would be formally given by:

$$\partial_t u = \frac{1}{2} \partial_x^2 u - \frac{1}{8} \frac{\mathrm{d}^2}{\mathrm{d}b^2} \, \mathrm{d}\ell_t^b(x) \Big|_{b=0} + \xi$$

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For the moment, no technique for solving such equations. Even for well-posed SPDEs, the existence of nice local time processes is not known in general: only partial results.

Related results and open questions

In 2005, Zambotti had already proved an IbPF for the law P_0^1 of the reflected Brownian motion on $[0,1]$. Setting $X_t := |B_t|, t \in [0,1]$, he showed :

$$E_0^1\left[\partial_h \Phi(X)\right] = -E_0^1\left[\langle h'', X\rangle \Phi(X)\right] + \mathbb{E}\left[\Phi(X)\int_0^1 h_r : \dot{B}_r^2 : d\ell_r^0\right],$$

where ℓ^0 is the local time at 0 of X.

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Here, the second term is actually a generalized functional on the Wiener space $C([0,1])$, defined via a "regularization-renormalization" procedure. This is reminiscent of the KPZ equation.

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Question: How are these formulae related with our IbPF ?

Idea of proof for $\Phi = 1$

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Suppose that $\Phi = 1$. Then $\partial_h \Phi = 0$, i.e. the first term vanishes.
For the second term, we have:

$$E^{\delta}_{a,a}\left[\langle h'', X\rangle \Phi(X)\right] = \int_0^1 \mathrm{d}r\, h''_r\, E^{\delta}_{a,a}[X_r] \\ = \int_0^1 \mathrm{d}r\, h_r\, \frac{\mathrm{d}^2}{\mathrm{d}r^2} E^{\delta}_{a,a}[X_r].$$

Note that, setting $x := a^2$, we have, for all $r \in (0,1)$:

$$E^{\delta}_{a,a}[X_r] = Q^{\delta}_{x,x}[\sqrt{Z_r}] = \int_0^{\infty} q^{\delta,x}_r(y)\sqrt{y}\,\mathrm{d}y,$$

where $q^{\delta,x}_r$ is the density of Z_r when $Z \overset{(d)}{=} Q^{\delta}_{x,x}$.

Equivalently:

$$E^{\delta}_{a,a}[X_r] = \Gamma\left(\frac{\delta+1}{2}\right)\left\langle \mu_{\frac{\delta+1}{2}}, \frac{q^{\delta,x}_r(y)}{y^{\delta/2-1}}\right\rangle.$$

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Regarding the last term: for $\delta \notin \{1,3\}$, performing the change of variable $y = b^2$, we have:

$$- \kappa(\delta) \int_0^1 \mathrm{d}r \; h(r) \int_0^\infty \mathrm{d}m_\delta(b)\, b^{-3}\, \mathcal{T}^{2k}_b \Sigma^a_{\delta,r}(\Phi(X)|\cdot) =$$

$$- \kappa(\delta) \int_0^1 \mathrm{d}r \; h(r) \int_0^\infty \mathrm{d}y\, y^{\frac{\delta-3}{2}-1}\, \mathcal{T}^{k}_y \left(\frac{q^{\delta,x}_r(y)}{y^{\delta/2-1}}\right) =$$

$$- \underbrace{\kappa(\delta)\Gamma\left(\frac{\delta-3}{2}\right)}_{\Gamma\left(\frac{\delta+1}{2}\right)} \int_0^1 dr \; h(r) \left\langle \mu_{\frac{\delta-3}{2}}, \frac{q^{\delta,x}_r(y)}{y^{\delta/2-1}} \right\rangle.$$

On the other hand, for $\delta = 3$, we have:

$$-\frac{1}{2}\int_0^1 \mathrm{d}r\, h(r)\, \Sigma^{a}_{3,r}(\Phi(X)|0) = -\int_0^1 \mathrm{d}r\, h(r) \left\langle \mu_0, \frac{q_r^{3,x}(y)}{y^{3/2-1}} \right\rangle,$$

while for $\delta = 1$, we have:

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h(r)\, \frac{\mathrm{d}^2}{\mathrm{d}b^2}\Sigma^{a}_{1,r}(\Phi(X)|b)\Big|_{b=0} = -\int_0^1 \mathrm{d}r\, h(r) \left\langle \mu_{-1}, \frac{q_r^{1,x}(y)}{y^{1/2-1}} \right\rangle.$$

Thus, whatever the value of $\delta > 0$, the last term in IbPF can be written as:

$$\Gamma\left(\frac{\delta+1}{2}\right)\int_0^1 \mathrm{d}r \; h(r) \left\langle \mu_{\frac{\delta-3}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}} \right\rangle$$

Thus, whatever the value of $\delta > 0$, the last term in IbPF can be written as:

$$\Gamma\left(\frac{\delta+1}{2}\right) \int_0^1 \mathrm{d}r \; h(r) \left\langle \mu_{\frac{\delta-3}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}} \right\rangle$$

So the IbPF for $\Phi = 1$ is equivalent to the relation :

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} \left\langle \mu_{\frac{\delta+1}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}} \right\rangle = \left\langle \mu_{\frac{\delta-3}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}} \right\rangle$$

holding for all $r \in (0,1)$ and $\delta > 0$.

Thus, whatever the value of $\delta > 0$, the last term in IbPF can be written as:

$$\Gamma\left(\frac{\delta+1}{2}\right)\int_0^1 \mathrm{d}r\ h(r)\left\langle \mu_{\frac{\delta-3}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}}\right\rangle$$

So the IbPF for $\Phi = 1$ is equivalent to the relation :

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left\langle \mu_{\frac{\delta+1}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}}\right\rangle = \left\langle \mu_{\frac{\delta-3}{2}}, \frac{q_r^{\delta,x}(y)}{y^{\delta/2-1}}\right\rangle$$

holding for all $r \in (0,1)$ and $\delta > 0$.
This can be checked by direct computations, using the properties of the distributions μ_α, $\alpha \in \mathbb{R}$.