A stochastic mass conserved reaction-diffusion equation with nonlinear diffusion

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Stochastic Partial Differential Equations

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Mass conserved Allen-Cahn equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta \varphi + f(\varphi) - \dfrac{1}{|D|} \displaystyle\int_D f(\varphi) dx, & x \in D, \quad t \geq 0, \\ \nabla \varphi . n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ \varphi(x, 0) = \varphi_0(x), & x \in D \end{cases}$$

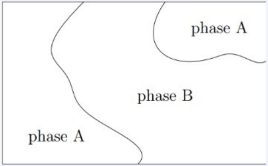
f has exactly 3 zeros $-1 < 0 < 1$ and

$$f'(\pm 1) < 0, \quad f'(0) > 0$$

A typical example is : $f(\varphi) = \varphi - \varphi^3$

Motivation

Deterministic mass conserved Allen-Cahn equation - linear diffusion.
Binary mixture undergoing phase separation.
[Rubinstein and Sternberg, 1992].



[Boussaïd,Hilhorst and Nguyen, 2015] proved the well-posedness and the stabilization of the solution for large times for the corresponding Neumann problem.

Stochastic Mass conserved Allen-Cahn equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta\varphi + f(\varphi) - \dfrac{1}{|D|}\displaystyle\int_D f(\varphi)dx + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ \nabla\varphi.n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ \varphi(x,0) = \varphi_0(x), & x \in D \end{cases}$$

Motivation

[Antonopoulou, Bates, Blömker and Karali, 2016]

- use the stochastic mass conserved equation with linear diffusion to describe the motion of a droplet.
- They study the singular limit of the solution of this equation, letting a small parameter tend to zero.
- They consider a white noise in time which also depends on space.

[Funaki and Yokoyama, 2016] study the singular limit of the solution of the stochastic mass conserved equation with linear diffusion with a smoothened one dimensional white noise in time.

Our goal today

Prove the existence and uniqueness of the solution of the nonlocal stochastic reaction-diffusion equation with nonlinear diffusion

$$(P) \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = div(A(\nabla \varphi)) + f(\varphi) - \dfrac{1}{|D|} \displaystyle\int_D f(\varphi) dx + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla \varphi).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ \varphi(x, 0) = \varphi_0(x), & x \in D \end{cases}$$

Even in the case when $A = I$ which they have been studying, no proof has been given by Antonopoulou, Bates, Blömker and Karali.

Hypotheses on the domain D and on the function A

- D open bounded set of $\mathbb{R}^n$, with a smooth boundary ∂D.
- A is such that
$$|A(a) - A(b)| \leq C|a - b|$$
- $A = \nabla\Psi : \mathbb{R}^n \to \mathbb{R}^n$ for some convex $C^{1,1}$-function Ψ and $A(0) = 0$.
- A is monotone
$$(A(a) - A(b))(a - b) \geq C_0(a - b)^2, \quad C_0 > 0$$
for all $a, b \in \mathbb{R}^n$.

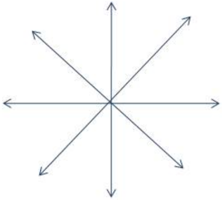
[Funaki, Spohn,1997].

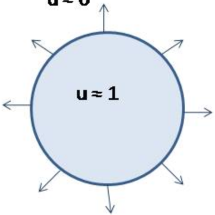
Remark:

$$\text{If } \; A = I \Rightarrow div(A(\nabla u)) = \Delta u.$$

Usual diffusion

Linear diffusion **(Isotropic equation) :**

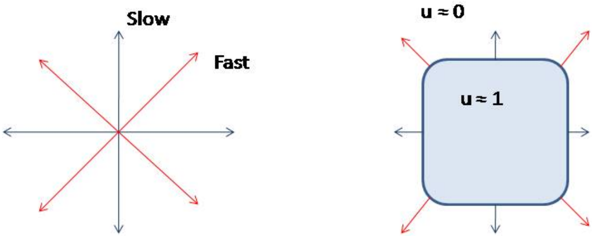




Diffusion and the speed are same in any direction.

Anisotropic diffusion

<u>Nonlinear diffusion</u> (Anisotropic equation) :



Diffusion and the speed depend on the directions.

The nonlinear function f

- The nonlinear function f is a smooth function which satisfies the following properties:

(F_1) There exist positive constants C_1 and C_2 such that

$$f(a+b)a \leq -C_1 a^{2p} + f_2(b), \quad |f_2(b)| \leq C_2(b^{2p}+1), \text{ for all } a, b \in \mathbb{R}$$

(F_2) There exist positive constants C_3 and $\tilde{C}_3(M)$ such that

$$|f(s)| \leq C_3|s-M|^{2p-1} + \tilde{C}_3(M)$$

(F_3) There exists a positive constant C_4 such that

$$f'(s) \leq C_4.$$

In particular, one can choose $f(s) = \sum_{j=0}^{2p-1} b_j s^j$ with $b_{2p-1} < 0, p \geq 2$, which also includes the Allen-Cahn equation with $f(s) = s(1-s^2)$.

The Q-Brownian motion

- The function $W(x,t)$ is a Q-Brownian motion in $L^2(D)$.

$$W(x,t) = \sum_{k=1}^{\infty} \beta_k(t) Q^{\frac{1}{2}} e_k(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t) e_k(x),$$

where

1. $\{e_k\}_{k\geq 1}$ is an orthonormal basis in $L^2(D)$ diagonalizing Q.
2. $\{\lambda_k\}_{k\geq 1}$ are the corresponding eigenvalues for all $k \geq 1$.
3. Q is a nonnegative definite symmetric operator on $L^2(D)$ with $\operatorname{Tr} Q < +\infty$.

$$\operatorname{Tr} Q = \sum_{k=1}^{\infty} \langle Q e_k, e_k \rangle_{L^2(D)} = \sum_{k=1}^{\infty} \lambda_k \leq \Lambda_0.$$

$$\sum_{k=1}^{\infty} \lambda_k \|e_k\|^2_{L^\infty(D)} \leq \Lambda_1 \tag{1}$$

4. $\{\beta_k(t)\}_{k\geq 1}$ is a sequence of independent $(\mathcal{F}_t)$-Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

The nonlinear stochastic heat equation

We consider the nonlinear stochastic heat equation

$$(P_1) \quad \begin{cases} \dfrac{\partial W_A}{\partial t} = div(A(\nabla W_A)) + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla(W_A)).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ W_A(x,0) = 0, & x \in D \end{cases}$$

[Krylov, Rozovskii, 2007]

Definition

1. $W_A \in L^\infty(0,T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T); H^1(D))$;
2. $div(A(\nabla W_A)) \in L^2(\Omega \times (0,T); (H^1(D))')$;
3. W_A *satisfies a.s. for a.e.* $t \in (0,T)$ *the problem*

$$\begin{cases} W_A(t) = \displaystyle\int_0^t div(A(\nabla W_A(s)))ds + W(t) \quad \textit{in } (H^1(D))', \\ A(\nabla W_A(t)).n = 0, \quad \textit{in the sense of distributions on } \partial D. \end{cases} \tag{2}$$

A preliminary change of functions

We define :

$$u(t) := \varphi(t) - W_A(t),$$

$$(P_2) \quad \begin{cases} \dfrac{\partial u}{\partial t} = div(A(\nabla(u+W_A)) - A(\nabla W_A)) + f(u+W_A) \\ \qquad\qquad -\dfrac{1}{|D|}\displaystyle\int_D f(u+W_A)dx, & x \in D, \quad t \geq 0, \\ A(\nabla(u+W_A)).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ u(x,0) = \varphi_0(x), & x \in D \end{cases}$$

Remark: The conservation of mass property holds, namely

$$\int_D u(x,t)dx = \int_D \varphi_0(x)dx, \text{ a.s. for a.e. } t \in \mathbb{R}^+.$$

Definition of the solution

Definition

1. $u \in L^\infty(0,T;L^2(\Omega\times D))\cap L^2(\Omega\times(0,T);H^1(D))\cap L^{2p}(\Omega\times(0,T)\times D)$, $div[A(\nabla(u+W_A))] \in L^2(\Omega\times(0,T);(H^1(D))')$;
2. *u satisfies the integral equation a.s. for a.e. $t \in (0,T)$ in the sense of distributions in D*

$$
\begin{aligned}
u(t) \quad &= \quad \varphi_0 + \int_0^t div[A(\nabla(u+W_A)) - A(\nabla W_A)]ds \\
&\quad + \int_0^t f(u+W_A) - \int_0^t \frac{1}{|D|}\int_D f(u+W_A)dxds
\end{aligned}
$$

3. *u satisfies the natural boundary condition in the sense of distributions on ∂D.*

Existence of a solution of Problem (P_2)

We work with the following spaces:

$$H = \left\{ v \in L^2(D), \int_D v = 0 \right\}, \quad V = H^1(D) \cap H \text{ and } Z = V \cap L^{2p}$$

Theorem

There exists a unique solution of Problem (P_2).

Proof:

We apply the Galerkin method, and use the following notations

- $0 < \gamma_1 < \gamma_2 \leq ... \leq \gamma_k \leq ...$ eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions.
- $w_k, k = 0, ...$ smooth unit eigenfunctions in $L^2(D)$.

Existence of a solution of Problem (P_2)

We recall that the functions $\{w_j\}$ are an orthonormal basis of $L^2(D)$, in particular:

$$\int_D w_j w_0 = 0 \quad \text{for all } j \neq 0 \;\; \text{and} \;\; w_0 = \frac{1}{\sqrt{|D|}},$$

which implies that $\int_D w_j(x)dx = 0$ for all $j \neq 0$.

Existence of a solution of Problem (P_2)

We look for an approximate solution of the form

$$u_m(x,t) - M = \sum_{i=1}^{m} u_{im}(t) w_i, \quad M = \frac{1}{|D|}\int_D \varphi_0(x)dx$$

which satisfies the equation:

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j$$
$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla w_j + \int_D f(u_m + W_A)w_j$$
$$-\frac{1}{|D|}\int_D \Big(\int_D f(u_m + W_A)dx\Big) w_j dx,$$

for all w_j, $j = 1, ..., m$.

$u_m(x,0) = M + \sum_{i=1}^{m}(\varphi_0, w_i)w_i$ converges strongly in $L^2(D)$ to φ_0 as $m \to \infty$.

Existence of a solution of Problem (P_2)

Remark: The contribution of the nonlocal term vanishes !!

$$\int_D w_j(x)dx = 0, \quad \text{for all } \; j \neq 0$$

$$\Downarrow$$

$$-\frac{1}{|D|}\int_D \Big(\int_D f(u_m + W_A)dx \Big) w_j = 0$$

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)w_j$$

$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla w_j + \int_D f(u_m + W_A)w_j$$

for all w_j, $j = 1, ..., m$.

A priori estimates

Lemma

There exists a positive constant $\mathcal{C}$ such that

$$
\begin{aligned}
\mathbb{E}\int_D (u_m(t)-M)^2 dx &\leq \mathcal{C}, \quad \textit{for all } t\in[0,T] \\
\mathbb{E}\int_0^T\int_D |\nabla(u_m-M)|^2 dxdt &\leq \mathcal{C} \\
\mathbb{E}\int_0^T\int_D (u_m-M)^{2p} dxdt &\leq \mathcal{C} \\
\mathbb{E}\int_0^T\int_D (f(u_m+W_A))^{\frac{2p}{2p-1}} &\leq \mathcal{C} \\
\mathbb{E}\int_0^T \|divA(\nabla(u_m+W_A))\|^2_{(H^1(D))'} &\leq \mathcal{C}
\end{aligned}
$$

A priori estimates

- Multiply the equation of u_m by $u_{jm} = u_{jm}(t)$ and sum on $j = 1, ..., m$

$$\int_D \frac{\partial}{\partial t}(u_m(x,t) - M)(u_m - M)$$

$$= -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla(u_m - M)$$

$$+ \int_D f(u_m + W_A)(u_m - M)$$

- Monotonicity property of A

$$
\begin{aligned}
& -\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(W_A))]\nabla(u_m - M) \\
& \leq -C_0 \int_D |\nabla(u_m - M)|^2
\end{aligned}
$$

- Using the property (F_1) we deduce that:

$$\begin{aligned}\int_D f(u_m + W_A(t))(u_m - M) &= \int_D f(u_m - M + M + W_A(t))(u_m - M) \\ &\leq -\int_D C_1(u_m - M)^{2p} + C_2 \int_D |W_A(t)|^{2p} \\ &\quad + \tilde{C}_2(M)|D|\end{aligned}$$

A priori estimates

Integrating from 0 to T and taking the expectation :

$$\mathbb{E}\int_D (u_m(T)-M)^2 dx + 2C_0\mathbb{E}\int_D |\nabla(u_m-M)|^2 dx$$
$$+2C_1\mathbb{E}\int_D (u_m-M)^{2p}$$
$$\leq \int_D (u_m(0)-M)^2 dx + 2C_2\mathbb{E}\int_0^T\int_D |W_A(t)|^{2p} + 2\tilde{C}_2(M)|D|T$$
$$\leq K$$

Weak convergence properties

Hence there exist a subsequence which we denote again by $\{u_m - M\}$ and functions
$u - M \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D) \cap L^\infty(0,T; L^2(\Omega \times D))$,
χ and Φ such that:

$$\begin{array}{rcll} u_m - M & \rightharpoonup & u - M & \text{weakly in } L^2(\Omega \times (0,T); V) \\ & & & \quad \text{and } L^{2p}(\Omega \times (0,T) \times D) \\ u_m - M & \rightharpoonup & u - M & \text{weakly star in } L^\infty(0,T; L^2(\Omega \times D)) \\ f(u_m + W_A) & \rightharpoonup & \chi & \text{weakly in } L^{\frac{2p}{2p-1}}(\Omega \times (0,T) \times D) \\ div(A(\nabla(u_m + W_A))) & \rightharpoonup & \Phi & \text{weakly in } L^2(\Omega \times (0,T); (H^1)') \end{array}$$

as $m \to \infty$.

Passing to the limit

$$\begin{aligned}\langle (u_m(x,t)-M), w_j\rangle &= \langle u_m(0)-M, w_j\rangle \\ &+\langle div[A(\nabla(u_m-M+W_A))-A(\nabla(W_A))], w_j\rangle \\ &+\langle f(u_m+W_A), w_j\rangle, \text{ for all } j=1,..,m.\end{aligned}$$

We pass to the limit as $m\to\infty$

$$\langle u(t)-M, w\rangle = \langle \varphi_0 - M, w\rangle + \int_0^t \langle \Phi - div(A(\nabla W_A)), w\rangle + \int_0^t \langle \chi, w\rangle$$

for all $w\in V\cap L^{2p}(D)$.

It remains to prove that :

$$\langle \Phi+\chi, w\rangle = \langle div(A(\nabla(u+W_A)))+f(u+W_A(t)), w\rangle \quad \text{for all } w\in V\cap L^{2p}(D).$$

Monotonicity argument

[Marion,1987], [Krylov,Rozovskii, 2007]

Let w be such that $w - M \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times D \times (0,T))$

$$\begin{aligned}\mathcal{O}_m &= \mathbb{E}\Big[\int_0^T e^{-cs}\{2\langle div\big(A(\nabla(u_m - M + W_A)) - A(\nabla W_A)\big) \\ &\quad -div\big(A(\nabla(w - M + W_A)) - A(\nabla W_A)\big), u_m - M - (w - M)\rangle_{Z^*,Z} \\ &\quad +2\langle f(u_m + W_A) - f(w + W_A), u_m - M - (w - M)\rangle_{Z^*,Z} \\ &\quad -c\|u_m - M - (w - M)\|^2\}ds \\ &= J_1 + J_2 + J_3\end{aligned}$$

Monotonicity argument

Lemma

$$O_m \leq 0$$

Proof.

- For the nonlinear diffusion term we use monotonicity

$$\begin{aligned} J_1 &= -2\mathbb{E}\int_0^T e^{-cs}\int_D [A(\nabla(u_m - M + W_A)) - A(\nabla(w - M + W_A))] \\ &\qquad [\nabla(u_m - M + W_A) - \nabla(w - M + W_A)] \\ &\leq -2C_0\mathbb{E}\int_0^T e^{-cs}\|\nabla(u_m - w)\|^2 \leq 0 \end{aligned}$$

□

Proof.

- For the reaction term we use the property (F_3)

$$\begin{aligned} J_2 &= \mathbb{E}\int_0^T e^{-cs} 2\langle f(u_m + W_A) - f(w + W_A), u_m - w\rangle_{Z^*,Z} ds \\ &\leq \mathbb{E}\int_0^T e^{-cs} 2C_4 \|u_m - w\|^2 ds. \end{aligned}$$

- The nonlocal term vanishes

Choosing $c \geq 2C_4$, we conclude the result. □

Let $v \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D)$ be arbitrary and set

$$w - M = u - M - \lambda v, \text{ with } \lambda \in \mathbb{R}_+.$$

Dividing by λ and letting $\lambda \to 0$, we find that :

$$\mathbb{E}\int_0^T e^{-cs}\langle \Phi + \chi - div(A\nabla(u - M + W_A) - f(u + W_A), v\rangle_{Z^*,Z} dt \leq 0$$

Since v is arbitrary, it follows that

$$\mathbb{E}\int_0^T \langle \Phi + \chi, v\rangle_{Z^*,Z} = \mathbb{E}\int_0^T \langle div[A(\nabla(u - M + W_A))] + f(u + W_A), v\rangle_{Z^*}$$

for all $v \in L^2(\Omega \times (0,T); V) \cap L^{2p}(\Omega \times (0,T) \times D)$,

Uniqueness

Proof:

- Let u_1 and u_2 be two solutions of Problem (P_2)

$$
\begin{aligned}
u_1(t) - u_2(t) &= \int_0^t div(A(\nabla(u_1 + W_A) - A(\nabla(u_2 + W_A)) \\
&+ \int_0^t [f(u_1 + W_A) - f(u_2 + W_A)] \\
&- \frac{1}{|D|} \int_0^t [\int_D f(u_1 + W_A) - \int_D f(u_2 + W_A)dx].
\end{aligned}
$$

Uniqueness

- Taking the duality product with $u_1 - u_2$
- Same initial condition $u_1(x,0) = u_2(x,0) = \varphi_0(x) \Rightarrow$
$$-\frac{1}{|D|}\int_0^t [\int_D f(u_1 + W_A) - \int_D f(u_2 + W_A)] \int_D (u_1 - u_2) = 0.$$
- Taking the expectation of the equation
$$\mathbb{E}\int_D (u_1 - u_2)^2(x,t)dx \leq C_4 \mathbb{E}\int_0^t \int_D (u_1 - u_2)^2(x,t)dxds,$$

By Gronwall's Lemma

$$u_1 = u_2 \quad \text{a.e. in } \Omega \times D \times (0,T).$$

The solution of the stochastic heat equation

Nonlinear stochastic heat equation

$$(P_1) \quad \begin{cases} \dfrac{\partial W_A}{\partial t} = div(A(\nabla W_A)) + \dfrac{\partial W}{\partial t}, & x \in D, \quad t \geq 0, \\ A(\nabla(W_A)).n = 0, & \text{on } \partial D \times \mathbb{R}^+ \\ W_A(x,0) = 0, & x \in D \end{cases}$$

[Krylov, Rozovskii, 2007]

Definition

1. $W_A \in L^\infty(0,T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T); H^1(D))$;
2. $div(A(\nabla W_A)) \in L^2(\Omega \times (0,T); (H^1(D))')$;
3. W_A *satisfies a.s. for a.e.* $t \in (0,T)$ *the problem*

$$\begin{cases} W_A(t) = \displaystyle\int_0^t div(A(\nabla W_A(s)))ds + W(t) \quad \textit{in } (H^1(D))', \\ A(\nabla W_A(t)).n = 0, \quad \textit{in the sense of distributions on } \partial D. \end{cases} \tag{3}$$

[Gess, 2012]

Definition

1. *W_A is a solution in the sense of Krylov and Rozovskii;*
2. $W_A \in L^2(\Omega; C([0,T]; L^2(D)))$;
3. $div(A(\nabla W_A)) \in L^2(\Omega \times (0,T); L^2(D))$;
4. *W_A satisfies a.s. for all $t \in (0,T)$ the problem*

$$\begin{cases} W_A(t) = \int_0^t div(A(\nabla W_A(s)))ds + W(t) \quad \text{in } L^2(D), \\ A(\nabla W_A(t)).n = 0, \quad \text{in a suitable sense of trace on } \partial D. \end{cases} \tag{4}$$

Note that

$$W_A \in L^2(\Omega \times (0,T); H^2(D))$$

L^q regularity

Theorem

Let W_A be a solution of Problem (P_1); then $W_A \in L^\infty(0, T; L^q(\Omega \times D))$, for all $q \geq 2$.

Proof.

For each positive constant k, denote by $\Phi_k : \mathbb{R} \to \mathbb{R}$ the function

$$\Phi_k(\xi) = \begin{cases} |\xi|^q, & \text{if } |\xi| < k, \\ \frac{q}{2}(q-1)k^{q-2}\xi^2 - q(q-2)k^{q-1}|\xi| + (\frac{q}{2} - 1)(q-1)k^q, \\ & \text{if } k \leq |\xi|. \end{cases}$$

Φ_k is a convex C^2 function and Φ_k' is a Lipschitz-continuous function with $\Phi_k'(0)$=0. The function Φ_k satisfies the inequalities $0 \leq \Phi_k'(\xi) \leq c(k)\xi$ and $0 \leq \Phi_k(\xi) = \int_0^\xi \Phi_k'(\zeta)d\zeta \leq \frac{c(k)}{2}\xi^2$ for all $\xi \in \mathbb{R}^+$. □

Inequalities for the test functions for L^q regularity

Lemma

1. *One has* $0 \leq \Phi_k''(\xi) \leq c_k$ *for all* $\xi \in \mathbb{R}$ *where* c_k *is a positive constant depending on* k.
2. *One has* $0 \leq \Phi_k''(\xi) \leq q(q-1)(1+\Phi_k(\xi))$, *for all* $\xi \in \mathbb{R}$.

Applying Itô's formula

$$\int_D \Phi_k(W_A(t))dx = \int_0^t \langle div(A(\nabla W_A(s))), \Phi_k'(W_A(s))\rangle ds + \int_0^t \int_D \Phi_k'(W_A(s))dW(s) + \frac{1}{2}\sum_{l=1}^{\infty} \int_0^t \int_D \Phi_k''(W_A)\lambda_l(e_l)^2 dxds$$

Using the monotonicity property of A

$$-\int_0^t \int_D \Phi_k''(W_A(s))\nabla W_A(s)A(\nabla W_A(s))ds \leq -C_0\mathbb{E}\int_0^t \int_D \Phi_k''(W_A)|\nabla W_A|^2$$

From (1) :

$$\frac{1}{2}\sum_{l=1}^{\infty}\int_0^t \int_D \Phi_k''(W_A)\lambda_l(e_l)^2 dxds \leq \frac{1}{2}\sum_{l=1}^{\infty}\lambda_l\|e_l\|_{L^\infty}^2 \int_0^t \int_D \Phi_k''(W_A)dxds$$

The solution of the stochastic heat equation

$$\begin{aligned}
\mathbb{E}\int_D \Phi_k(W_A(t))dx &\leq -C_0\mathbb{E}\int_0^t\int_D \Phi_k''(W_A)|\nabla W_A|^2 \\
&\quad +\frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k''(W_A)dxds \\
&\leq \frac{1}{2}q(q-1)\Lambda_1\mathbb{E}\int_0^t\int_D (1+\Phi_k(W_A))dxds \\
&\leq C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}
\end{aligned}$$

Since $\Phi_k(W_A(x,t))$ converges to $|W_A(x,t)|^q$ for a.e. x and t when k tends to infinity

$$\mathbb{E}\int_D |W_A(x,t)|^q dx = \mathbb{E}\int_D \lim_{k\to\infty}\Phi_k(W_A(x,t))dx \leq C(q)\Lambda_1 t|D|e^{C(q)\Lambda_1 t}$$

for all $t > 0$.

Therefore, $W_A \in L^\infty(0,T;L^q(\Omega\times D))$ for all $q \geq 2$.

THANK YOU FOR YOUR ATTENTION