Stochastic homogenization of the Landau-Lifshitz equation

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Stochastic Partial Differential Equations
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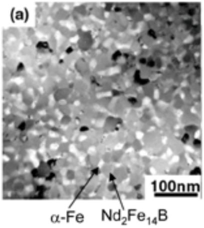
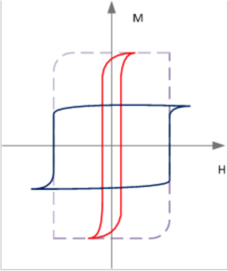
Exchange spring magnets

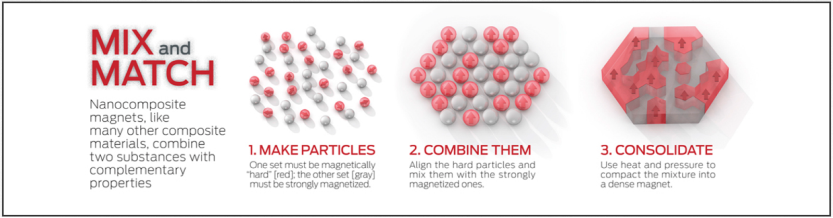




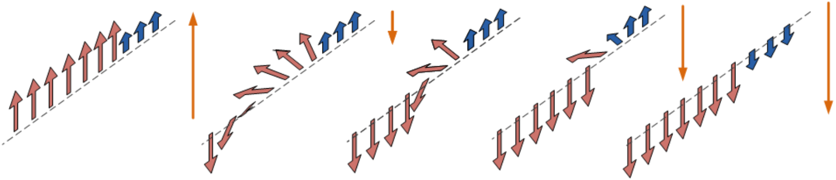
"This is a new type of magnet comprising a fine sub-micron hard magnetic phase and soft magnetic phase that behaves like a homogenous and uniform magnet with magnetic exchange coupling operation working between the two phases... It is said that if a high-coercivity exchange spring magnet with anisotropy could be developed, it would be the best magnet in the world, but this goal has not been attained at this time. "

(Source : http ://www.shinetsu-rare-earth-magnet.jp)

Exchange spring magnets

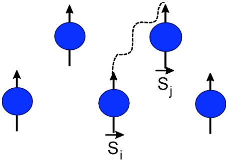


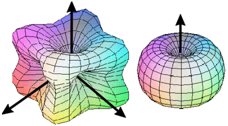
(source : *G. Hadjipanayis and A. Gabay.The incredible pull of nanocomposite magnets. IEEE Spectrum*)

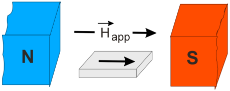


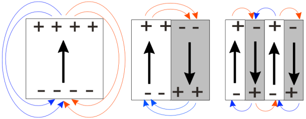
(source : Wikipedia)

Brown energy









$$
\begin{aligned}
\mathcal{E}(m) = &\int_D a_{ex} |\nabla m|^2 \\
&+ \int_D k(1 - (m \cdot u)^2) \\
&- \mu_0 \int_D H_{ext} \cdot M_s m \\
&- \frac{\mu_0}{2} \int_{\mathbb{R}^3} H_d(M_s m) \cdot M_s m
\end{aligned}
$$

Dynamical equation

Landau-Lifshitz $\sim$ 1935 :

$$H_{eff}(m) = -D_m\mathcal{E}(m) = \mathrm{div}\,(a_{ex}\nabla m) + l.o.t$$

then the LL equation is given by

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_{eff}(m) - \alpha m \times (m \times H_{eff}(m)) \ \text{ in } D \\ m(0,x) = m_0(x) \ \text{ in } D \\ \dfrac{\partial m}{\partial n} = 0 \ \text{ on } \partial D \end{cases}$$

$\alpha > 0$, damping :

$$\frac{d\mathcal{E}(m)}{dt} = -\int_D H_{eff}(m)\cdot\frac{\partial m}{\partial t}\,dx = -\alpha\int_D |m \times H_{eff}(m)|^2\,dx$$

The homogenization problem

- The material is made of several compounds (with extra asumptions on the compounds distribution)
- Each compound has its own a_{ex}, M_S, k, u
- The size of each compound is $\varepsilon << 1$

$\rightsquigarrow$ understand the limit as $\varepsilon \to 0$ of the model with

$$\mathcal{E}_\varepsilon(m) = \int_D a_{ex}\left(\frac{x}{\varepsilon}\right)|\nabla m|^2 + \int_D k\left(\frac{x}{\varepsilon}\right)\left(1 - \left(m \cdot u\left(\frac{x}{\varepsilon}\right)\right)^2\right)$$
$$-\frac{\mu_0}{2}\int_{\mathbb{R}^3} H_d\left(M_s\left(\frac{x}{\varepsilon}\right)m\right)\cdot M_s\left(\frac{x}{\varepsilon}\right)m - \mu_0\int_D H_{ext}\cdot M_s\left(\frac{x}{\varepsilon}\right)m$$

that is : convergence of $\min_{H^1(D,S^2)} \mathcal{E}_\varepsilon(m)$ as $\varepsilon \to 0$? of stationary solutions ? of dynamical solutions ?

Reminders about homogenization (periodic case)

Model problem

$$\begin{cases} -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } D \\ u_\varepsilon = 0 \text{ on } \partial D \end{cases} \tag{1}$$

- "multiscale expansion"

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

with $u_1(x,y)$ Q-periodic in y, so that

$$\nabla_x u_\varepsilon(x) \sim \nabla_x u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

- Plugging the ansatz leads to 2 problems

i) The **cell problem** : $\forall x \in D$

$$\begin{cases} -\mathrm{div}_y(a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))) = 0 \text{ for } y \in Q \\ \text{with } u_1(x,\cdot)\ Q\text{- periodic} \end{cases}$$

Note that

$$u_1 = \sum_{k=1}^{3} \partial_{x_k} u_0 \; \chi_k$$

where the correctors χ_k solves

$$-\mathrm{div}_y(a(y)\nabla_y \chi_k) = \mathrm{div}_y(a(y)e_k)$$

with (e_1, e_2, e_3) basis of $\mathbf{R}^3$. There is a unique Q-periodic solution χ_k with $\int_Q \chi_k dy = 0$.

ii) The **homogenized problem**

$$\begin{cases} -\mathrm{div}_x\Big(\displaystyle\int_Q a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))\, dy\Big) = f \text{ for } y \in D \\ u_0(x) = 0 \text{ on } \partial D \end{cases}$$

which may be written as

$$-\sum_{x_i} \partial_i \Big[\sum_j \int_Q a_{i,j}(y)(\delta_{j,k} + \partial_{y_j}\chi_k)\, dy \Big] \partial_{x_k} u_0 = f$$

Hence, if

$$A_{hom} = \int_Q a(y)(I + \nabla_y \chi)\, dy,$$

- u_0 solves

$$\begin{cases} -\mathrm{div}_x(A_{hom} \nabla u_0) = f \text{ for } y \in D \\ u_0(x) = 0 \text{ on } \partial D \end{cases}$$

- For any $\xi \in \mathbf{R}^3$,

$$A_{hom}\xi \cdot \xi = \inf_{\phi \in H^1_\sharp(Q)} \int_Q a(y)(\xi + \nabla\phi(y)) \cdot (\xi + \nabla\phi(y))\, dy$$

- u_ε minimizes in $H^1_0(D)$

$$\frac{1}{2}\int_D a\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla u\, dx - \int_D fu\, dx$$

- u_0 minimizes in $H^1_0(D)$

$$\frac{1}{2}\int_D A_{hom} \nabla u \cdot \nabla u\, dx - \int_D fu\, dx$$

Periodic homogenization of the Brown energy

Alouges, Di Fratta, 2015 ; let a_{ex} be $(0,1)^3$-periodic and let

$$\mathcal{E}_\varepsilon(m) = \int_D a_{ex}\left(\frac{x}{\varepsilon}\right) |\nabla m|^2, \quad \text{with } m(x) \in S^2$$

i.e. consider for simplicity only the exchange energy. Then

Theorem : $\mathcal{E}_\varepsilon(m)$ Γ-converges in the weak H^1 topology to $\int_D Tg_{hom}(m, \nabla m)\, dx$ where, for any ξ,

$$Tg_{hom}(m,\xi) = \inf_{\phi \in H^1_\sharp(Q, m^\perp)} \int_Q a_{ex}(y)|\xi + \nabla\phi(y)|^2\, dy\,.$$

Morever, for any $m \in S^1$, for any ξ,

$$\begin{aligned} Tg_{hom}(m,\xi) = g_{hom}(\xi) &= \inf_{\phi \in H^1_\sharp(Q, \mathbb{R}^3)} \int_Q a_{ex}(y)|\xi + \nabla\phi(y)|^2\, dy \\ &= A_{hom}\xi \cdot \xi \end{aligned}$$

independent of $m \rightsquigarrow g_{hom}$ is the **classical** homogenization density.

Stochastic homogenization of the LLG equation

In spring magnets, the structure is not periodic but randomly distributed

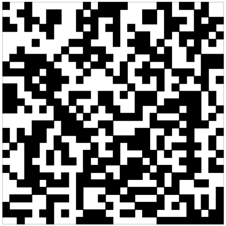


Figure 1 A piece of a sample of a random checkerboard. The conductivity matrix is equal to $\mathbf{a}_0$ in the black region, and $\mathbf{a}_1$ in the white region.

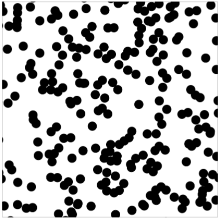


Figure 2 A sample of the coefficient field defined in (0.9) by the Poisson point cloud. The matrix $\mathbf{a}$ is equal to $\mathbf{a}_0$ in the black region and to $\mathbf{a}_1$ in the white region.

S. Armstrong, T. Kuusi and J.-C. Mourrat, Quantitative stochastic homogenization and large-scale regularity, arXiv

Consider the Landau-Lifshitz equation :

$$\begin{cases} \dfrac{\partial m_\varepsilon}{\partial t} & = & m_\varepsilon \times \operatorname{div}\big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon\big) - \alpha m_\varepsilon \times \big(m_\varepsilon \times \operatorname{div}\big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon\big)\big) \\ m_\varepsilon(0) & = & m_0, \quad x \in D \end{cases}$$

with $|m_0(x)| = 1$, a.e. in D.

Here a is a random field (possibly matrix valued) indexed by $\mathbf{R}^3$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

or the harmonic maps equation :

$$\begin{cases} \operatorname{div}\big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon\big) \times m_\varepsilon(x) = 0 \\ |m_\varepsilon(x)| = 1, \ a.e. \text{ on } D. \end{cases}$$

Note that the later equation is equivalent to

$$\mathrm{div}\Big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon\Big) = |\nabla m_\varepsilon|^2 m_\varepsilon$$

(due to the constraint $|m_\varepsilon| = 1$, a.e.)

Weak formulation : $m_\varepsilon \in H^1(D, \mathbf{R}^3)$, $|m_\varepsilon| = 1, a.e.$

$$\int_D \Big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon(x) \times m_\varepsilon(x)\Big) \cdot \nabla\phi(x)\, dx = 0$$

for all regular test functions ϕ.

Existence of weak solutions : Chen, Coron et al., $\sim$ 1990 (stationary or heat flow) ; **No uniqueness**

Landau-Lifshitz-Gilbert equation

Weak formulation : (using the equivalent Gilbert form)
$m_\varepsilon \in H^1((0,T)\times D)$, $|m_\varepsilon| = 1$, a.e. and for all regular test functions ϕ on $(0,T)\times D = Q_T$,

$$\int_{Q_T} \Big(\frac{\partial m_\varepsilon}{\partial t} + \alpha m_\varepsilon \times \frac{\partial m_\varepsilon}{\partial t}\Big) \cdot \phi \, dxdt$$
$$= (1+\alpha^2) \int_{Q_T} \Big(a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon \times m_\varepsilon\Big) \cdot \nabla\phi \, dxdt.$$

Moreover, $m_\varepsilon(0) = m_0$ in the trace sense and there is a $K > 0$ such that

$$\frac{1}{2}\int_D a(\frac{x}{\varepsilon},\omega)\nabla m_\varepsilon \cdot \nabla m_\varepsilon dx + \int_0^t \int_D \Big|\frac{\partial m_\varepsilon}{\partial t}\Big|^2 dxdt \le K$$

Existence of weak solutions : Visintin, 1985, Alouges-Soyeur, 1989. **No uniqueness**

The random structure

Kozlov ; Papanicolaou-Varadhan ; Piatnitskii Zhikov (stochastic two-scale convergence), ...

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space (canonical space) ; $\mathbf{P} = \mathcal{L}(a)$
- $(T_x)_{x \in \mathbf{R}^3}$ translation group on Ω ; assume $\mathbf{P}$ is T-invariant (stationarity), and T si ergodic
- $a(x, \omega) = a(T_x \omega)$ is assumed to be stochastically continuous $\rightsquigarrow$ Ω is a compact metric space
- $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$, and $C(\Omega)$, are separable
- $D_i u(\Omega) = \frac{\partial}{\partial x_i} u(T_x \omega)_{|x=0}$ infinitesimal generator of translations along the 3 axis ; closed and densely defined operator ; $\mathcal{H}^1 = \cap_{i=1}^3 \mathcal{D}(D_i)$ is densely defined and separable
- $L^2_{pot}(\Omega)$: closure of $\{D_\omega u,\ u \in C^1(\Omega)\}$ in $L^2(\Omega; \mathbf{R}^3)$.

Diffusion equation :

G. Papanicolaou and S. Varadhan, 1979.

$$\begin{cases} -\mathrm{div}\left(a\left(T_{\frac{x}{\varepsilon}}\omega\right)\nabla u_\varepsilon(x,\omega)\right) = f \text{ in } D\times\Omega \\ u_\varepsilon = 0 \text{ on } \partial D\times\Omega \end{cases} \tag{2}$$

Then, $(u_\varepsilon)_\varepsilon$ strongly converges in $L^2(D\times\Omega)$ towards the solution of

$$\begin{cases} -\mathrm{div}\left(A_{s,hom}\nabla u_0(x)\right) = f \text{ in } D \\ u_0 = 0 \text{ on } \partial D \end{cases} \tag{3}$$

where, for all $\xi \in \mathbf{R}^3$,

$$\begin{aligned} A_{s,hom}\xi\cdot\xi &= \inf_\psi \mathbb{E}\left(a(\omega)|\xi + D\psi(\omega)|^2\right) \\ &= \inf_\psi \int_\Omega a(\omega)|\xi + D\psi(\omega)|^2 d\mathbb{P}(\omega) \end{aligned}$$

Stochastic 2-scale convergence

Periodic 2-scale convergence

Nguetseng, 1989, Allaire, 1992 : $m_\varepsilon(x) \twoheadrightarrow m_0(x, y)$ if for all smooth test function ϕ on $D \times Q$, y-periodic,

$$\int_D m_\varepsilon(x)\phi(x, x/\varepsilon)\, dx \to \int_{D\times Q} m_0(x, y)\phi(x, y)\, dx\, dy$$

Stochastic 2-scale convergence :

Piatnitskii, Zhikov, 2006 ; (Bourgeat, Mikelic, Wright, 1994 : in mean) : $m_\varepsilon(x) \twoheadrightarrow m_0(x, \omega)$ if for all "smooth" ϕ on $D \times \Omega$, for a.e. $\omega_0 \in \Omega$,

$$\int_D m_\varepsilon(x)\phi(x, T_{x/\varepsilon}\omega_0)\, dx \to \int_{D\times\Omega} m_0(x, \omega)\phi(x, \omega)\, dx\, d\mathbb{P}(\omega)$$

Pbe : Choice of ω_0 ?

Birkhoff ergodic theorem

For all $\phi \in L^1(\Omega)$,

$$\lim_{t \to +\infty} \frac{1}{t^3|A|} \int_{tA} \phi(T_x \omega_0)\, dx = \int_\Omega \phi(\omega) d\mathbb{P}(\Omega) = \mathbb{E}(\phi)$$

for $\mathbb{P}$ a.e. $\omega_0 \in \Omega$.

However : With previous assumptions, one may prove that for some Ω_0 with $\mathbb{P}(\Omega_0) = 1$, for all $\omega_0 \in \Omega_0$, and for all $\phi \in C(\Omega)$,

$$\lim_{\varepsilon \to 0} \int_A \phi(T_{x/\varepsilon}\omega_0)\, dx = \mathbb{E}(\phi).$$

Ω_0 : typical trajectories

Stochastic 2-scale convergence

Def : $\omega_0 \in \Omega_0$ being fixed, $m_\varepsilon(x) \twoheadrightarrow m_0(x,\omega)$ if for all $\phi \in C_c^\infty(\mathbf{R}^3)$, and all $\psi \in C(\Omega)$,

$$\int_D m_\varepsilon(x)\phi(x)\psi(T_{x/\varepsilon}\omega_0)\,dx \to \int_{D\times\Omega} m_0(x,\omega)\phi(x)\psi(\omega)\,dx\,d\mathbb{P}(\omega)$$

Theorem :

There exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$, such that if $\omega_0 \in \Omega_0$ and if $(m_\varepsilon)_\varepsilon$ is bounded in L^2 then there exists $m_0 \in L^2(D \times \Omega)$ s.t. up to a subsequence

$$m_\varepsilon \twoheadrightarrow m_0$$

stochastically two scale.

Homogenization of the LLG equation

Theorem : Alouges, dB, Merlet, Nicolas, 2018

Let $\omega_0 \in \Omega_0$, let $m_\varepsilon(t) \in H^1(Q_T; S^2)$ be a family of weak solutions of (LLG) (resp. a family of weak Harmonic maps such that $(m_\varepsilon)_\varepsilon$ is bounded in $H^1(D)$) then, up to extraction, m_ε weakly converges to $\bar{m}$ in $H^1((0, T) \times D)^3$, which is a weak solution of

$$\begin{cases} \dfrac{\partial \bar{m}}{\partial t} = \bar{m} \times \operatorname{div}\left(A_{s,hom} \nabla \bar{m}\right) - \alpha \bar{m} \times \left(\bar{m} \times \operatorname{div}(A_{s,hom} \nabla \bar{m})\right) \\ \bar{m}(0) = m_0 \end{cases}$$

(resp. m_ε weakly converges in $H^1(D, S^2)$ to $\bar{m}$ which is a weak solution of

$$-\operatorname{div}\left(\bar{m} \times A_{s,hom} \nabla \bar{m}\right) = 0)$$

and $A_{s,hom}$ is the classical stochastic homogenization tensor.

Ideas of proof (harmonic maps)

- Use Rellich Theorem, and the previous theorem $\rightsquigarrow$ up to extraction :
$$m_\varepsilon \to \bar{m} \text{ strongly in } L^2$$
$$\nabla m_\varepsilon \rightharpoonup \nabla_x \bar{m} + \xi(x,\omega),$$
weakly in L^2, where $\xi \in L^2_{pot}(\Omega)$.
- We recover $|\bar{m}| = 1$, a.e. and it follows $\bar{m} \perp \xi_i$, for $i = 1, 2, 3$.
- Use the variational formulation :
$$\int_D \Big(a(T_{x/\varepsilon}\omega_0) m_\varepsilon(x) \times \nabla m_\varepsilon(x) \Big) \cdot \phi(x) dx = 0$$
with $\phi(x) = \varepsilon \psi(x) v(T_{x/\varepsilon}\omega_0)$; $\psi \in C_c^\infty(D)$; $v \in C^1(\Omega, \mathbf{R}^3)$
$\rightsquigarrow$ using weak-strong convergence
$$\int_\Omega a(\cdot)(\partial_j \bar{m}(x) + \xi_j(x,\cdot)) \cdot D_j(v \times \bar{m}(x))\, d\mathbb{P} = 0.$$

This shows that $a(.)(\nabla \bar{m}(x) + \xi)$ is in the $L^2(\Omega)$-orthogonal of $L^2_{pot}(\Omega, \mathbf{R}^3)$ and thus

$$\int_\Omega a(\omega)(\nabla \bar{m}(x) + \xi(x,\omega))d\mathbb{P}(\omega) = A_{s,hom}\nabla \bar{m}(x)$$

i.e ξ is a solution of the cell problem.

Next, take $\phi \in C_c^\infty(D, \mathbf{R}^3)$ and taking the limit implies

$$\int_D \bar{m}(x) \times \Big(\int_\Omega a(\omega)(\nabla_x \bar{m}(x) + \xi(x,\omega) d\mathbb{P}(\omega)\Big) \cdot \phi(x)\, dx = 0$$

which leads to

$$-\mathrm{div}(\bar{m} \times A_{s,hom}\nabla \bar{m}) = 0.$$

LLG : works in the same way, adapting the definition of the stochastic two-scale convergence (take account of time dependence).

Remark on the energy inequality :

Usually, for weak solution of LLG, one has the energy inequality :

$$\frac{1}{2}\int_D a(x)\nabla m(t)\cdot\nabla m(t)dx + \int_0^t\int_D \left|\frac{\partial m}{\partial t}\right|^2 dxds \\ \leq \frac{1}{2}\int_D a(x)\nabla m_0\cdot\nabla m_0 dx.$$

However, we only have for the homogenized solution :

$$\frac{1}{2}\int_D A_{s,hom}\nabla\bar{m}(t)\cdot\nabla\bar{m}(t)\,dx + \int_0^t\int_D \left|\frac{\partial \bar{m}}{\partial t}\right|^2 dxds \\ \leq \int_D \mathbb{E}(a)|\nabla\bar{m}(0)|^2 dx.$$

Other energy terms

When all energy terms are taken into account, on must write $m^\varepsilon(t,x) = M^\varepsilon(x)u^\varepsilon(t,x)$, with $|u^\varepsilon| = 1$, and u^ε and M^ε both depend on the material component ; then u^ε satisfies the LL equation :

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = m \times H^\varepsilon_{eff}(u^\varepsilon) - \alpha u^\varepsilon \times (u^\varepsilon \times H^\varepsilon_{eff}(u^\varepsilon)) \text{ in } D \\ u^\varepsilon(0,x) = u_0(x) \text{ in } D \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } \partial D \end{cases}$$

with

$$H^\varepsilon_{eff}(u) = \mathrm{div}(a^\varepsilon \nabla u) + M^\varepsilon h_d(M^\varepsilon u) + M^\varepsilon h_{ext} - \nabla_u \phi_\varepsilon(u).$$

Assume :

- $a^\varepsilon(x,\omega) = a(T_{x/\varepsilon}\omega)$ (same assumptions as before)
- $M^\varepsilon(x,\omega) = M_S(T_{x/\varepsilon}\omega)$, M_S in $C(\Omega)$ and bounded
- $\phi_\varepsilon(x,u) = \phi_{an}(T_{x/\varepsilon}\omega, u)$ with $\nabla_u \phi_{an}$ in $C(\Omega)$; typically

$$\phi_{an}(\omega, u) = \kappa(\omega)(1 - (v(\omega)\cdot u)^2),$$

v along the easy axis, $|v| = 1$, κ is bounded

Then, up to a subsequence, u^ε converges weakly in H^1 to a weak solution of the LL equation with

$$\begin{aligned} H^0_{eff}(u) \quad = \quad & \mathrm{div}(A_{s,hom}\nabla u) + \mathbb{E}(M_S)h_d(M_S u) + \mathbb{E}(M_S P_{L^2_{pot}}(M_S u)) \\ & + \mathbb{E}(M_S)h_{ext} + \nabla_u \mathbb{E}(\phi_{an}(u)). \end{aligned}$$

Conclusion

- Unified formalism for periodic and stochastic homogenization 2-scale convergence (Piatniskii and Zhidkov)
- No need for uniqueness of the solution to the limiting equation
- Could also obtain $\Gamma-$convergence type results for stochastic case
- Stochastic homogenization for harmonic maps equation
- Stochastic homogenization of Landau-Lifshitz equation (including all physical terms)
- Numerical implementation + optimization of the distribution...