Global solutions to reaction-diffusion equations with super-linear drift and multiplicative noise

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Based on joint work with:

Davar Khoshnevisan (Univ. of Utah), Tusheng Zhang (Univ. of Manchester)

$$\frac{\partial}{\partial t}u(t\,,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t\,,x) + b(u(t\,,x)) + \sigma(u(t\,,x))\,\dot{W}(t\,,x), \qquad (1)$$

$$t > 0, \qquad x \in]0\,,1[,$$

$\dot{W}$ is space-time white noise,

homogeneous Dirichlet boundary conditions:

$$u(t\,,0) = u(t\,,1) = 0 \qquad \text{for all } t > 0,$$

initial condition: $u(0\,,x) = u_0(x)$, $x \in [0\,,1]$.

$$b, \sigma : \mathbb{R} \to \mathbb{R}$$

Classical case: b and σ are globally Lipschitz (implies linear growth).

Krylov & Rozovskii (1979), Walsh (1986), ...

Here: b and σ will be locally Lipschitz with superlinear growth.

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Related literature on spde's with coefficients with superlinear growth

Mueller (1991). Considers $b(u) \equiv 0$, $\sigma(u) = u^\gamma$, with $1 \leqslant \gamma < \frac{3}{2}$ (locally Lipschitz, superlinear growth). Establishes existence of a global solution.

Krylov (1996). Considers (essentially) $b(u) = u$, $|\sigma(u)| \leqslant c|u|^\gamma$, with $1 \leqslant \gamma < \frac{3}{2}$. Establishes existence of a global solution.

Mueller & Sowers (1993). $b \equiv 0$, $\sigma(u) = u^\gamma$. Establish that for γ large enough, there is blowup in finite time.

Mueller (2000). $b \equiv 0$, $\sigma(u) = u^\gamma$, $\gamma > \frac{3}{2}$. There exists $\tau < \infty$ a.s. such that

$$P\left\{\lim_{t\uparrow\tau} \sup_{x\in[0,1]} u(t,x) = +\infty\right\} > 0.$$

Monotonicity conditions: Donati-Martin & Pardoux (1993), Cerrai (2003), Liu & Röckner (2015).

Typical case: $b(u) = -u^3$ (locally Lipschitz, but pushes back towards origin).

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Deterministic case: $\sigma \equiv 0$

Case of o.d.e.'s:

$$u'(t) = b(u(t)), \qquad u(0) = u_0, \qquad b : \mathbb{R} \to \mathbb{R}_+.$$

Then finite-time blowup occurs if and only if Osgood's condition holds:

$$\int_{u_0}^{+\infty} \frac{dz}{b(z)} < +\infty$$

($b \uparrow +\infty$ quickly enough for the integral to be finite).

Example: $b(u) = u^\gamma$ with $\gamma > 1$. Then finite-time blowup occurs.

Deterministic case: $\sigma \equiv 0$, continued

Deterministic heat equation: $\sigma \equiv 0$, $b(u) = u^\gamma$, $\gamma > 1$:

$$\frac{\partial}{\partial t} u(t\,,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t\,,x) + (u(t\,,x))^\gamma, \qquad (2)$$

$$x \in \,]0,1[, \qquad t > 0,$$

Dirichlet b.c.

$$u(0\,,x) = c\,u_0(x), \text{ where } u_0 \geqslant 0,\ u_0 \not\equiv 0,\ u_0 \in C([0,1],\mathbb{R}_+),\ c > 0.$$

- $\gamma \in \,]1,3[$: all classical solutions blow up in finite time.
- $\gamma > 3$:
  - for c large enough, all classical solutions blow up in finite time.
  - for c small enough, there exist small global solutions (for c small, u^γ is small, and the solution stays near 0).
  - there exist small stationary solutions.

Reference: Galaktionov & Vasquez (2002).

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Bonder & Groisman (2009)

Corollary. The deterministic heat equation (2) may have a global solution even in the case where

$$\int_1^{+\infty} \frac{dz}{b(z)} < +\infty$$

For instance, this occurs if $b(z) = z^\gamma$ and $\gamma > 3$.

Theorem 1 (Bonder & Groisman, 2009)

Suppose $\sigma \equiv \sigma_0 \neq 0$, $b \geqslant 0$, b is convex and

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Then finite-time blowup occurs in the SPDE (1), that is, there exists $\tau < +\infty$ a.s. such that

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Question

Is the Bonder & Groisman result optimal?

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$$\int_1^{+\infty} \frac{dz}{b(z)} = +\infty,$$

is there finite-time blowup?

For example, what happens for $b(z) = |z| \log_+ |z|$?

Definition

A random field solution $u = \{u(t,x),\ t \geqslant 0,\ x \in [0,1]\}$ *of the stochastic heat equation* (1) *is a jointly measurable, adapted process such that, for all* $(t,x) \in \mathbb{R}_+ \times [0,1]$,

$$u(t,x) = (G_t * u_0)(x) + \int_{[0,t]\times[0,1]} G_{t-s}(x,y)\, b(u(s,y))\, dsdy + \int_{[0,t]\times[0,1]} G_{t-s}(x,y)\, \sigma(u(s,y))\, W(ds,dy),$$

where $G_t(x,y)$ *is the heat kernel with zero Dirichlet boundary conditions.*

Remark

The stochastic integral is a "localized Walsh integral," that is, we only require that

$$\int_{[0,t]\times[0,1]} [G_{t-s}(x,y)\, \sigma(u(s,y))]^2\, dsdy\ \infty \qquad a.s.,$$

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Notion of solution to the stochastic heat equation

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Main result

Notation. For $\alpha \in]0\,,1]$,

$$\mathbb{C}_0^\alpha := \Big\{ f : [0\,,1] \to \mathbb{R} \text{ such that } f(0) = f(1) = 0 \text{ and } \|f\|_{\mathbb{C}_0^\alpha} := \sup_{0 \le x < y \le 1} \frac{|f(y) - f(x)|}{|y - x|^\alpha} < \infty \Big\}.$$

Theorem 2 (D., Khoshnevisan & Zhang (2017))

Suppose that:

i. $u_0 \in \bigcup_{0<\alpha\le 1} \mathbb{C}_0^\alpha$;

ii. *b and σ are locally Lipschitz functions such that*

$$|b(z)| = O(|z| \log |z|) \quad \text{and} \quad |\sigma(z)| = o\left(|z|(\log |z|)^{1/4}\right), \quad \text{as } z \to \pm\infty.$$

Then the stochastic heat equation (1) has a random field solution u in $C(\mathbb{R}_+ \times [0,1])$ (global solution), and this solution is unique. In particular, u satisfies

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} |u(t\,,x)| < \infty \qquad \text{a.s. for all } T \in]0\,,\infty[.$$

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Case where b and σ are globally Lipschitz

Notation. For a globally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, there are constants $c(f)$ and $L(f)$ such that

$$|f(z)| \leqslant c(f) + \mathrm{L}(f)|z|, \qquad \text{for all } z \in \mathbb{R}.$$

Remark. One possible choice for $\mathrm{L}(f)$ is the Lipschitz constant $\|f\|_{\mathbb{C}_0^1}$ of f. In this case, one can take $c(f) = |f(0)|$. Often, the smallest possible value of $\mathrm{L}(f)$ is smaller than $\|f\|_{\mathbb{C}_0^1}$.

Proposition

Let u be the solution of (1). There exists $A < +\infty$ such that, for all $t \geqslant 0$, for all $k \in [2, \sqrt{\mathrm{L}(b)}/\mathrm{L}(\sigma)^2]$,

$$\sup_{x\in[0,1]} \mathbb{E}\left(|u(t,x)|^k\right) \leq \left[A\left(\|u_0\|_{L^\infty} + \frac{c(b)}{\mathrm{L}(b)} + \frac{c(\sigma)}{\sqrt{\mathrm{L}(\sigma)}}\right) \cdot \exp\left(A\,\mathrm{L}(b)\,t\right)\right]^k \qquad (3)$$

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Proof of Proposition

(u_n): sequence of Picard iterates used to construct the solution of (1). Define

$$\mathcal{N}_{\beta,k}(u_n) := \sup_{t\geq 0} \sup_{x\in[0,1]} \left(e^{-\beta t} \|u_n(t\,,x)\|_k \right).$$

By a direct calculation:

$$\mathcal{N}_{\beta,k}(u_{n+1}) \leq C_{\beta,k} + L_{\beta,k}\, \mathcal{N}_{\beta,k}(u_n), \tag{4}$$

where

$$L_{\beta,k} := c \max\left(\frac{\mathrm{L}(b)}{\beta} \,,\; \frac{k^{1/2}\mathrm{L}(\sigma)}{\beta^{1/4}} \right)$$

and $C_{\beta,k}$ is the constant on the r.h.s. of (3).

Choose $\beta = 16c^4\mathrm{L}(b)$. Then

$$L_{16c^4\mathrm{L}(b),k} \leqslant \max\left(\frac{1}{16} \,,\; \frac{k^{1/2}\mathrm{L}(\sigma)}{2[\mathrm{L}(b)]^{1/4}} \right) \leqslant \frac{1}{2} \tag{5}$$

since $2 \leqslant k \leqslant \sqrt{\mathrm{L}(b)}/\mathrm{L}(\sigma)^2$. Then (4) and (5) imply:

$$\mathcal{N}_{16c^4\mathrm{L}(b),k}(u) \leqslant \limsup_{n\to\infty} \mathcal{N}_{16c^4\mathrm{L}(b),k}(u_n) \leqslant 2C_{\beta,k}.$$

The Proposition is proved with $A = 16c^4$.

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Uniform moment bounds

Let u be the solution of (1).

Theorem 3

Suppose $u_0 \in \mathbb{C}_0^\alpha$, $\alpha \in]0,1]$. *Let* $w = \max(12, 6/\alpha)$, *fix* $T_0 > 0$.
Assume that $\sqrt{\mathrm{L}(b)} > w\mathrm{L}(\sigma)^2$.
Then there exists $A < \infty$ *such that, for all* $T \in]0\,, T_0]$, *for all*
$k \in \left]w\,, \sqrt{\mathrm{L}(b)}/\mathrm{L}(\sigma)^2\right]$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\sup_{x\in[0,1]}|u(t\,,x)|^k\right) \qquad (6)$$

$$\leq \left[A(1\vee T)^{(1+\frac{\alpha}{2}\wedge\frac{1}{4})}\left(\|u_0\|_{\mathbb{C}_0^\alpha} + k^{1/2}\mathcal{M}_1 + k^{1/2}\mathcal{M}_2\mathcal{M}_3\,\mathrm{e}^{A\,\mathrm{L}(b)\,T}\right)\right]^k,$$

where

$$\mathcal{M}_1 := c(b) + c(\sigma); \quad \mathcal{M}_2 := \mathrm{L}(b) + \mathrm{L}(\sigma); \quad \mathcal{M}_3 := \|u_0\|_{L^\infty} + \frac{c(b)}{\mathrm{L}(b)} + \frac{c(\sigma)}{\mathrm{L}(\sigma)}.$$

Idea of proof of Theorem 3

By the Dirichlet boundary condition,

$$|u(t,x)| = |u(t,x) - u(t,0)|.$$

Do a careful estimate of $E[|u(t,x) - u(s,y)|^k]$, keeping track of Lipschitz constants:

$$E[|u(t,x) - u(s,y)|^k] \leqslant \left[A\left(\|u_0\|_{\mathbb{C}_0^\alpha} + k^{1/2}\mathcal{M}_1 + k^{1/2}\mathcal{M}_2\mathcal{M}_3\,\mathrm{e}^{A\,\mathrm{L}(b)\,T}\right)\right]^k \times \left(|t-s|^{\frac{\alpha}{2}\wedge\frac{1}{4}} + |x-y|^{\alpha\wedge\frac{1}{2}}\right)^k$$

In order to get the supremum inside the expectation, use an anisotropic Kolmogorov Continuity Theorem [D., Khoshnevisan & E. Nualart (2007)]:

$$E\left[\sup_{\substack{(t,x)\neq(s,y)\\ s\leqslant t\leqslant T}} \frac{|u(t,x)-u(s,y)|^k}{\left[|t-s|^{\frac{\alpha}{2}\wedge\frac{1}{4}} + |x-y|^{\alpha\wedge\frac{1}{2}}\right]^{k\delta-w}}\right] \leqslant \text{r.h.s. of (6)}$$

for $\delta \in \left]\frac{w}{k}, 1\right[$.

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Recall the statement:

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Suppose that $u_0 \in \cup_{0<\alpha\leq 1}\mathbb{C}_0^\alpha$, and b and σ are locally Lipschitz functions such that

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Then the stochastic heat equation (1) *has a unique (global solution).*

Proof. Define the truncation of the function $b(z)$:

$$b_N(z) := \begin{cases} b(-N) & \text{if } z < -N, \\ b(z) & \text{if } |z| \leq N, \\ b(N) & \text{if } z > N. \end{cases}$$

$\sigma_N(z)$ is defined similarly. Then b_N and σ_N are globally Lipschitz.

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Ideas in the proof of Theorem 2 (continued)

In the spde, replace b by b_N and σ by σ_N:

$$\frac{\partial}{\partial t} u_N(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_N(t,x) + b_N(u_N(t,x)) + \sigma(u_N(t,x))\,\dot{W}(t,x), \qquad (7)$$

with same b.c. and i.c. Then (7) has globally Lipschitz coefficients.

Suppose (for simplicity) that $b(z) = \theta_1 + \theta_2 |z| \log_+ |z|$. Then

$$L(b_N) = \theta_2(1 + \log N), \qquad L(\sigma_N) = o((\log N)^{1/4}).$$

In particular,

$$\frac{\sqrt{\mathrm{L}(b_N)}}{(\mathrm{L}(\sigma_N))^2} \underset{N \to +\infty}{\longrightarrow} +\infty.$$

Define

$$\tau_N^1 = \inf\{t \geqslant 0 : \sup_{x \in [0,1]} |u_N(t,x)| > N\}.$$

Will show:

$$\tau_\infty^1 := \lim_{N \to \infty} \tau_N^1 = +\infty \qquad \text{a.s.}$$

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In the spde, replace b by b_N and σ by σ_N:

$$\frac{\partial}{\partial t} u_N(t\,,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_N(t\,,x) + b_N(u_N(t\,,x)) + \sigma(u_N(t\,,x))\,\dot{W}(t\,,x), \qquad (7)$$

with same b.c. and i.c. Then (7) has globally Lipschitz coefficients.

Suppose (for simplicity) that $b(z) = \theta_1 + \theta_2 |z| \log_+ |z|$. Then

$$L(b_N) = \theta_2(1 + \log N), \qquad L(\sigma_N) = o((\log N)^{1/4}).$$

In particular,

$$\frac{\sqrt{\mathrm{L}(b_N)}}{(\mathrm{L}(\sigma_N))^2} \underset{N\to+\infty}{\longrightarrow} +\infty.$$

Define

$$\tau_N^1 = \inf\{t \geqslant 0 : \sup_{x\in[0,1]} |u_N(t,x)| > N\}.$$

Will show:

$$\tau_\infty^1 := \lim_{N\to\infty} \tau_N^1 = +\infty \qquad \text{a.s.}$$

Ideas in the proof of Theorem 2 (continued)

Fix $\varepsilon > 0$. Then

$$P\{\tau_N^1 < \varepsilon\} = P\left\{\sup_{t\in[0,\varepsilon]}\sup_{x\in[0,1]} |u_N(t,x)| > N\right\}.$$

By Chebychev, this is

$$\leqslant \frac{1}{N^k} E\left[\sup_{t\in[0,\varepsilon]}\sup_{x\in[0,1]} |u_N(t,x)|^k\right] \qquad (k > w = \max(12, \tfrac{6}{\alpha}))$$

By Theorem 3, this is

$$\leqslant \frac{1}{N^k}\left[A\|u_0\|_{\mathbb{C}_0^\alpha}(B+\log N)e^{A\varepsilon\log N}\right]^k \equiv C\,(B+\log N)^k\, N^{k(A\varepsilon-1)}$$

$$\longrightarrow 0 \qquad \text{as } N \to +\infty, \text{ if } \varepsilon = \varepsilon_0 > 0 \text{ is small enough.}$$

Therefore, $\tau_\infty^1 \geqslant \varepsilon_0 > 0$ a.s.

Ideas in the proof of Theorem 2 (continued)

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Therefore, non-explosion is guaranteed for $\varepsilon_0 > 0$ units of time.

By the Markov property, can restart at time ε_0: non-explosion is guaranteed for $2\varepsilon_0$ units of time, ..., $k\varepsilon_0$ units of time, for all k.

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Another formulation of long-term existence

L^2–initial condition: $u_0 \in L^2[0,1] =: \mathbb{L}^2$
Consider $\mathbb{L}^2_{loc}$-*solutions*: solutions up to a stopping time τ (variational formulation of (1)).

Theorem 4

Suppose in addition that $\sigma : \mathbb{R} \to \mathbb{R}$ is bounded, and $|b(z)| = O(|z| \log |z|)$ as $|z| \to \infty$. Then, every $\mathbb{L}^2_{loc}$-solution u of (1) is a long-time solution:

$$\sup_{t \in [0, \tau \wedge T]} \int_0^1 |u(t,x)|^2 \, dx < \infty \qquad \text{a.s. for every } T \in [0, \infty[.$$

Remark

(a) Suppose τ is a maximal time up to which the solution can be constructed:

$$\sup_{t \in [0, \tau[} \|u(t)\|_{L^2[0,1]} = \infty \qquad \text{a.s.}$$

Then Theorem 4 implies that $\tau = \infty$ a.s.: $\sup_{t \in [0,T]} \|u(t)\|_{L^2[0,1]} < \infty$ a.s. for all $T > 0$.
(b) The question of the existence of an $\mathbb{L}^2_{loc}$-solution of (1) under the assumptions of Theorem 4 is open.

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Ideas in the proof of Theorem 4

Define

$$\tau(R) := \begin{cases} \inf\{t \in [0, \tau[: \|u(t)\|_{\mathbb{L}^2} > R\} & \text{if } \{\cdots\} \neq \varnothing, \\ \tau & \text{otherwise.} \end{cases}$$

To prove that $P\{\sup_{t<\tau\wedge T} \|u(t)\|_{\mathbb{L}^2} = \infty\} = 0$, it suffices to prove that

$$\lim_{R\to\infty} P\{\tau(R) < \tau \wedge T\} = 0 \qquad \text{for all } T > 0. \tag{8}$$

Define

$$v_R(t, x) = \int_{[0,t]\times[0,1]} G_{t-s}(x, y)\sigma(u(s \wedge \tau(R), y))\, W(ds\, dy),$$

and

$$d_R := u - v_R,$$

so that on $\{\tau > t\}$,

$$\dot{d}_R(t) = \tfrac{1}{2} d_R''(t) + b\,(v_R(t) + d_R(t))\,, \tag{9}$$

subject to initial condition $d_R(0) = u_0$ and Dirichlet b.c.

Ideas in the proof of Theorem 4 (continued)

By the Kolmogorov continuity theorem, v_R has a jointly continuous version. Define

$$\tau_M(R) := \inf\left\{t > 0 : \sup_{x\in[0,1]} |v_R(t\,,x)| > M\right\} \qquad M > 0.$$

Then

$$\sup_{R>0} P\left\{\tau_M(R) < T\right\} \leq \sup_{R>0} \frac{\mathbb{E}(\sup_{t\in[0,T]} \sup_{x\in[0,1]} |v_R(t\,,x)|)}{M} \leqslant \frac{A_T}{M}. \qquad (10)$$

Define

$$D(t) := d_R\left(t \wedge \tau(R) \wedge \tau_M(R)\right), \quad V(t) := v_R\left(t \wedge \tau(R) \wedge \tau_M(R)\right) \qquad [0 \leq t \leq T],$$

and the Lyapunov function [Fang & Zhang (2005)]

$$\Phi(r) := \exp\left(\int_0^r \frac{dz}{1 + z\log_+ z}\right) \qquad [r > 0].$$

Notice that

$$\Phi'(r)[1 + r\log_+ r] = \Phi(r) \qquad \text{for all } r \geq 0. \qquad (11)$$

Ideas in the proof of Theorem 4 (continued)

By the Chain Rule:

$$\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) = \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) + \int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right) \frac{d}{ds}\|D(s)\|_{\mathbb{L}^2}^2$$

Since

$$\frac{d}{ds}\|D(s)\|_{\mathbb{L}^2}^2 = 2\,\langle \dot{D}(s)\,,\,D(s)\rangle_{\mathbb{L}^2} = 2\,\langle \tfrac{1}{2}D''(s) + b(V(s)+D(s))\,,\,D(s)\rangle_{\mathbb{L}^2},$$

we get

$$\begin{aligned}\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) &= \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) - \int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right) \|D'(s)\|_{\mathbb{L}^2}^2\,ds \\ &\quad + 2\int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right) \langle b(V(s)+D(s))\,,D(s)\rangle_{\mathbb{L}^2}\;ds. \qquad (12)\end{aligned}$$

From the $L\log L$ growth of b, can deduce

$$\langle b(V(s)+D(s))\,,D(s)\rangle_{\mathbb{L}^2} \le \bar{C}(b,M)\,\left\{\|D(s)\|_{\mathbb{L}^2\log\mathbb{L}}^2 + \|D(s)\|_{\mathbb{L}^2}^2 + 1\right\}.$$

Consequence of Gross' log-Sobolev inequality:

$$\|h\|_{\mathbb{L}^2\log\mathbb{L}}^2 \le \varepsilon\|h'\|_{\mathbb{L}^2}^2 + K_\varepsilon\|h\|_{\mathbb{L}^2}^2 + \|h\|_{\mathbb{L}^2}^2\log_+\left(\|h\|_{\mathbb{L}^2}^2\right) + \mathrm{e}^{-1},$$

with $\varepsilon := 1/(2\bar{C}(b,M))$ to get

$$\langle b(V(s)+D(s))\,,D(s)\rangle_{\mathbb{L}^2} \le \tfrac{1}{2}\|D'(s)\|_{\mathbb{L}^2}^2 + c_*\left\{\|D(s)\|_{\mathbb{L}^2}^2 + \|D(s)\|_{\mathbb{L}^2}^2\log_+\left(\|D(s)\|_{\mathbb{L}^2}^2\right) + 1\right\}.$$

Ideas in the proof of Theorem 4 (continued)

Combine with (12) to get rid of the term with $D'(s)$:

$$\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) \leq \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) + C^* \int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right)\left\{1 + \|D(s)\|_{\mathbb{L}^2}^2 \log_+\left(\|D(s)\|_{\mathbb{L}^2}^2\right)\right\} ds.$$

Combine with the property (11) of Φ to get

$$\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) \leq \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) + C \int_0^t \Phi\left(\|D(s)\|_{\mathbb{L}^2}^2\right) ds, \qquad (13)$$

From Gronwall's inequality:

$$\sup_{R>0} \mathbb{E}\left[\Phi\left(\|d_R(T \wedge \tau(R) \wedge \tau_M(R)\|_{\mathbb{L}^2}^2\right)\right] \leq C(b, M, T). \qquad (14)$$

Conclusion

Finally, we prove (8):

$$
\begin{aligned}
&P\{\tau(R) < \tau \wedge T\} \\
&\qquad \leqslant P\{\tau(R) < \tau \wedge T \leq \tau_M(R)\} + P\{\tau_M(R) < T\} \\
&\qquad \leqslant P\left\{\Phi\left(\|d(T \wedge \tau(R) \wedge \tau_M(R))\|_{\mathbb{L}^2}^2\right) \geq \Phi\left((R-M)^2\right)\right\} + \frac{A_T}{M} \qquad \text{(by (10))} \\
&\qquad \leqslant \frac{C(b, M, T)}{\Phi\left((R-M)^2\right)} + \frac{A_T}{M}. \qquad \text{(by (14))}
\end{aligned}
$$

Let $R \to \infty$, then $M \to \infty$, to obtain (8).
Theorem 4 is proved. $\square$

Conclusion. The Bonder-Groisman condition is essentially optimal!

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