Path regularity of the solution to the stochastic heat equation with Lévy noise

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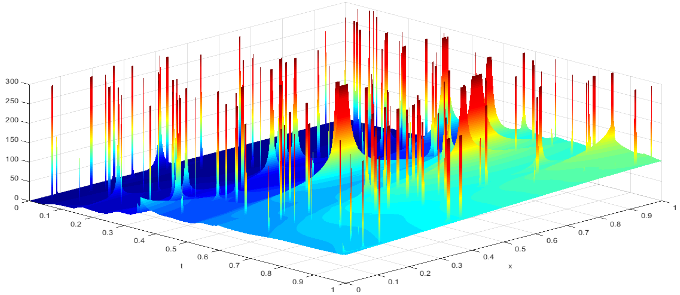
Marseille, May 2018

Joint work with Robert Dalang and Thomas Humeau (both EPFL)



Stochastic heat equation with multiplicative Lévy noise

$$\begin{cases} (\partial_t - \Delta)u(t,x) = \sigma(u(t,x))\dot{L}(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

- σ globally Lipschitz continuous function
- u_0 continuous bounded initial condition
- $\dot{L}$ **Lévy space–time white noise**

Lévy noise

Lévy space–time white noise

$$L = (L(A)\colon A \text{ is a bounded Borel subset of } \mathbb{R}_+ \times \mathbb{R}^d)$$

- L is an $L^0(\Omega)$-valued random measure.
- $(L(A_i))_{i\in\mathbb{N}}$ are independent for pairwise disjoint $(A_i)_{i\in\mathbb{N}}$
- The law of $L(A)$ only depends on $\mathrm{Leb}(A)$

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- ~~$L(A)$ is $N(0, \mathrm{Leb}(A))$-distributed~~

Why non-Gaussian noise?

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$$\mathbb{P}[X_1 > x] \sim a x^{-\alpha}, \quad x \to \infty, \quad (\mathbb{E}[X_1^2] = \infty!)$$

for some $\alpha \in (0, 2)$, then

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \xrightarrow{d} S\alpha S(v_a)$$

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2. **Poisson limit theorem:** If $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{P}[X_1 = 1] = 1 - \mathbb{P}[X_1 = 0] = \lambda/n$, then

$$\sum_{i=1}^{n} X_i \xrightarrow{d} \textbf{Poisson}(\lambda)$$

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The Gaussian assumption is questionable if ...

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Applications:

- **Physics:** Energy dissipation in turbulence
- **Biology:** Neuron potentials
- **Finance**: Interest rates

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$$\text{Discrete noise } \dot{W}_\epsilon \quad \xrightarrow{\text{CLT}} \quad \text{Continuous noise } \dot{W}$$

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- Convergence of noise/solution should not be measured in C^α topologies.
- Topology should be "punish" jumps too much!
- This is different to SDEs with jumps: approximation holds in uniform topology!

Theorem: Lévy–Itô decomposition (Adler et al. 1983)

Every Lévy space–time white noise can be decomposed as

$$L(A) = \underbrace{b\mathrm{Leb}(A)}_{\text{drift}} + \underbrace{cW(A)}_{\text{Gaussian part}} + \underbrace{\int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_A(s,y) z \mathbf{1}_{|z|\geq 1}\, \mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)}_{\text{big jumps part}=L^P(A)}$$

$$+ \underbrace{\int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_A(s,y) z \mathbf{1}_{|z|< 1}\, (\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)}_{\text{small jumps part}=L^M(A)}$$

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μ: **Poisson random measure**, i.e. $\mu = \sum_{i=1}^{\infty} \delta_{(S_i,Y_i,Z_i)}$

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ν: **intensity measure** of μ:

$$\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) = \mathrm{d}s\,\mathrm{d}y\,m(\mathrm{d}z)$$

where m is the **Lévy measure** of L, i.e.:

$$\int_{\mathbb{R}} (1\wedge|z|^2)\,m(\mathrm{d}z) < \infty.$$

Some examples and properties

Examples

- **Symmetric α-stable noise** with $\alpha \in (0, 2)$ if

$$b = c = 0, \quad m(\mathrm{d}z) = a|z|^{-1-\alpha}\,\mathrm{d}z$$

- **Standard Poisson noise** if

$$b = c = 0, \quad m(\mathrm{d}z) = \delta_1(\mathrm{d}z)$$

Some examples and properties

Moments: For $q \in (0, \infty)$ we have

$$\int_{\mathbb{R}} |z|^q \mathbf{1}_{|z| \geq 1} \, m(\mathrm{d}z) < \infty \iff \mathbb{E}[|L(A)|^q] < \infty \text{ for bounded } A$$

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α-stable noise with $\alpha \in (0, 2)$

- **Infinitely many** jumps on any space–time domain!
- $\boldsymbol{\alpha} < \mathbf{1}$: Jumps summable $\Longrightarrow$ $L(\omega; \mathrm{d}t, \mathrm{d}x)$ is a measure!
- $\boldsymbol{\alpha} \geq \mathbf{1}$: Jumps non-summable $\Longrightarrow$ $L(\omega; \mathrm{d}t, \mathrm{d}x) =$ is **not** a measure!

Stochastic heat equation with Lévy noise

Heat equation with Lévy noise (mild formulation):

$$u(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u(s,y))\, L(\mathrm{d}s,\mathrm{d}y) \tag{SHE-L}$$

where L is a Lévy space–time white noise **without Gaussian part**.

Existence and uniqueness

Theorem (C. 2017)

If the Lévy measure m satisfies

$$\int_{|z|\leq 1} |z|^p \, m(\mathrm{d}z) + \int_{|z|>1} |z|^q \, m(\mathrm{d}z) < \infty$$

with some

$$0 < p < 1 + \frac{2}{d} \quad \text{and} \quad q > \frac{p}{1 + (1 + \frac{2}{d} - p)}$$

then (SHE-L) admits a mild solution u.

Previous results: Existence and uniqueness if

$$\int_{\mathbb{R}} |z|^p \, m(\mathrm{d}z) < \infty.$$

See Saint Loubert Bié (1998).

Problem formulation

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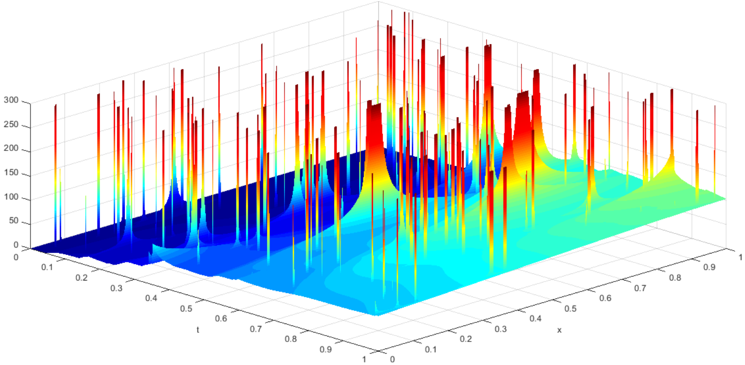

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- If the noise has infinite activity, there are infinitely many singularities on any non-empty open subset
- The paths $(t, x) \mapsto u(t, x)$ cannot be continuous, càdlàg, of bounded p-variation etc.

What kind of path properties?

Ideas:

- Regularity $t \mapsto u(t, \cdot)$ as a process with values in an **infinite dimensional space**

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- Regularity $t \mapsto u(t, \cdot)$ as a process with values in an **infinite dimensional space**

- Partial regularity of
  - $t \mapsto u(t, x)$ for fixed x
  - $x \mapsto u(t, x)$ for fixed t

A negative result

Theorem: see e.g. Peszat and Zabczyk (2007)

With additive Lévy noise, the process $t \mapsto u(t, \cdot)$ does not have a càdlàg modification in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$.

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Why?

- Every jump contributes $G(t - T_i, x - X_i)Z_i$ to the solution.
- $G(\epsilon, \cdot) \to \delta_0$ as $\epsilon \to 0$.
- $\delta_0 \notin L^p_{\mathrm{loc}}(\mathbb{R}^d)$.

Sobolev spaces of real order

Definition

The **Sobolev space of order** $r \in \mathbb{R}$ is given by

$$H_r(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \colon \xi \mapsto (1+|\xi|^2)^{\frac{r}{2}} \mathcal{F}(f)(\xi) \in L^2(\mathbb{R}^d) \right\},$$

equipped with the norm

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We also define the localized spaces

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Important example

For $x \in \mathbb{R}^d$, δ_x belongs to $H_r(\mathbb{R}^d)$ if and only if $r < -d/2$.

Càdlàg modifications

Theorem (C., Dalang & Humeau 17)

Under the existence conditions above, the process $t \mapsto u(t, \cdot)$ has a càdlàg modification in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for every $r < -d/2$.

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Previous results: Kotelenez (1982), Hausenblas (2009), Zhu, Brzeźniak and Hausenblas (2009)

Novelty: Heavy tailed noise (e.g. α-stable noise)

Partial regularity

What about the regularity of

$$t \mapsto u(t, x), \quad x \mapsto u(t, x)$$

for fixed x or fixed t, respectively?

Regularity in space for fixed time

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1. Suppose $\int_{|z|\leq 1} |z|^p \, m(\mathrm{d}z) < \infty$ for some $p \in (0, \frac{2}{d} \wedge 2)$.

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2. Suppose that $\sigma = 1$ and m behaves like $|z|^{-\alpha-1}\,\mathrm{d}z$ around the origin with some $\alpha \in [\frac{2}{d}, 2)$.

   Then for every $t \in [0, T]$, the section $x \mapsto u(t,x)$ is almost surely unbounded on every non-empty open subset of $\mathbb{R}^d$.

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Thank you very much!