The weak-A_∞ condition for harmonic measure

Xavier Tolsa

April 23, 2018

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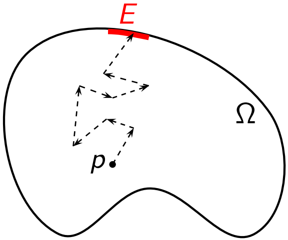
Probabilistic interpretation [Kakutani]:
When Ω is bounded, $\omega^p(E)$ is the probability that a particle with a Brownian movement leaving from $p \in \Omega$ escapes from Ω through E.



Rectifiability

We say that $E \subset \mathbb{R}^d$ is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.

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E is *n*-AD-regular if

$$\mathcal{H}^n(B(x,r)\cap E) \approx r^n \qquad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

E is uniformly *n*-rectifiable if it is *n*-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

Metric properties of harmonic measure

- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^1 \approx \omega^p$. (F.& M. Riesz)
- Many results in $\mathbb{C}$ using complex analysis (Makarov, Jones, Bishop, Wolff,...).
- The analogue of Riesz theorem fails in higher dimensions (counterexamples by Wu and Ziemer).
- In higher dimension, need real analysis techniques. Connection with uniform rectifiability studied recently by Hofmann, Martell, Uriarte-Tuero, Mayboroda, Azzam, Badger, Bortz, Toro, Akman, etc.
- A basic result of Dahlberg: If Ω is a Lipschitz domain, then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.
- What happens in more general domains?

Uniform, semiuniform, and NTA domains

Let $\Omega \subset \mathbb{R}^{n+1}$ be open.

- For $x, y \in \overline{\Omega}$, a curve $\gamma \subset \overline{\Omega}$ from x to y is a C-cigar curve with bounded turning if
  - $\min(\mathcal{H}^1(\gamma(x,z)), \mathcal{H}^1(\gamma(y,z))) \leq C \operatorname{dist}(z, \Omega^c)$, and
  - $\mathcal{H}^1(\gamma) \leq C\,|x-y|$.

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$$\text{NTA} \quad \subsetneq \quad \text{uniform} \quad \subsetneq \quad \text{semiuniform.}$$

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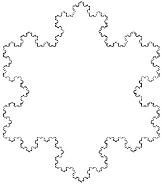
A non trivial NTA domain:



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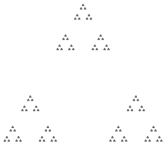
Example: The complement of this Cantor set is uniform but not NTA:



Harmonic measure in different types of domains

Definition: We say that $\omega \in A_\infty$ if, for any ball B centered in $\partial\Omega$ and $p \in \Omega \setminus 2B$, $\omega^p \in A_\infty(\mathcal{H}^n|_{\partial\Omega\cap B})$ uniformly.

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Theorem (David, Jerison / Semmes)

If Ω is NTA and $\partial\Omega$ is uniformly n-rectifiable, then $\omega \in A_\infty$.

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- A previous partial result by Aikawa and Hirata.

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Theorem (Hofmann, Le)

Let $\Omega \subset \mathbb{R}^{n+1}$, with $\partial\Omega$ n-AD-regular, satisfying the interior corkscrew condition. TFAE:

(a) *For some $p > 1$, the Dirichlet problem is L^p-solvable, i.e.*

$$\|Nu\|_{L^p(\mathcal{H}^n|_{\partial\Omega})} \leq C\, \|f\|_{L^p(\mathcal{H}^n|_{\partial\Omega})} \quad \text{for all } f \in L^p(\mathcal{H}^n|_{\partial\Omega}).$$

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(b) *The Dirichlet problem is BMO-solvable, i.e. for any ball B centered at $\partial\Omega$,*

$$\int_{B\cap\Omega} |\nabla u|^2 \,\text{dist}(x,\partial\Omega)\,dx \leq C\,\|f\|^2_{BMO(\mathcal{H}^n|_{\partial\Omega})}\, r(B)^n.$$

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(c) $\omega \in \mathrm{weak}{-}A_\infty$.

Remarks

- Ω satisfies the interior corkscrew condition if for every ball B centered at $\partial\Omega$ with $r(B) \leq \text{diam}(\Omega)$ there is another ball $B' \subset B \cap \Omega$ with $r(B') \approx r(B)$.

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- We say that $\omega \in \mathrm{weak}{-}A_\infty$ if for every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that for every ball B centered at $\partial\Omega$, all $p \in \Omega \setminus 4B$, and all $E \subset B \cap \partial\Omega$, the following holds:

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- Problem: Find a geometric characterization of the weak$-A_\infty$ condition.

Geometric characterization of the weak$-A_\infty$ condition I

- $\omega \in \mathrm{weak}{-}A_\infty$ + interior corkscrew condition $\Longrightarrow$ $\partial\Omega$ is uniformly n-rectifiable [Hofmann, Martell], [Mourgoglou-T.].

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- The uniform n-rectifiability of $\partial\Omega$ can be characterized in terms of a corona type decomposition for harmonic measure (Garnett-Mourgoglou-T.).

Geometric characterization of the weak$-A_\infty$ condition II

- Given $x \in \Omega$, $y \in \partial\Omega$, a c-carrot curve from x to y is a curve $\gamma \subset \Omega \cup \{y\}$ with end-points x and y such that $\text{dist}(z, \partial\Omega) \geq c\,\mathcal{H}^1(\gamma(y,z))$ for all $z \in \gamma$, where $\gamma(y,z)$ is the arc in γ between y and z.

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- We denote $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$.

- We say that Ω satisfies the weak local John condition if there are $\lambda, \theta \in (0,1)$ such that for every $x \in \Omega$ there is a Borel set $F \subset B(x, 2\delta_\Omega(x)) \cap \partial\Omega$ with $\mathcal{H}^n(F) \geq \theta\,\mathcal{H}^n(B(x, 2\delta_\Omega(x)) \cap \partial\Omega)$ such that every $y \in F$ can be joined to x by a λ-carrot curve.

Our main result I

Theorem (Hofmann, Martell)

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with uniformly n-rectifiable boundary satisfying the weak local John condition. Then $\omega \in \mathrm{weak}{-}A_\infty$.

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Theorem (Azzam, Mourgoglou, T.)

Let $\Omega \subset \mathbb{R}^{n+1}$ be open with n-AD-regular boundary. If $\omega \in \mathrm{weak}{-}A_\infty$, then Ω satisfies the weak local John condition.

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Putting all together:

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Remark

Very recently Hofmann and Martell have shown that (b) $\Rightarrow$ Ω has big pieces of chord-arc subdomains (BPCAS).

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Putting all together:

Theorem

Let $\Omega \subset \mathbb{R}^{n+1}$ be open with n-AD-regular boundary, satisfying the interior corkscrew condition. TFAE:

(a) $\omega \in \mathrm{weak}{-}A_\infty$.

(b) $\partial\Omega$ *is uniformly n-rectifiable and* Ω *satisfies the weak local John condition.*

Remark

Very recently Hofmann and Martell have shown that (b) $\Rightarrow$ Ω has big pieces of chord-arc subdomains (BPCAS).

Since BPCAS $\Rightarrow \omega \in \mathrm{weak}{-}A_\infty$ (Bennewitz, Lewis), we have

$$\text{BPCAS} \iff (a) \iff (b).$$

Some ideas for the proof

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  Otherwise we could argue with different poles $p_1, p_2, ...$
- Let $\mu = \mathcal{H}^n|_{\partial\Omega}$. We consider the good set G of points $x \in \partial\Omega \cap B(p, 2\delta_\Omega(p))$ such that

$$\omega^p(B(x, r)) \approx \frac{1}{\delta_\Omega(p)^n} \mu(B(x, r)).$$

  By the weak-A_∞ property, $\mu(G) \approx \mu(B(p, 2\delta_\Omega(p)) \approx \delta_\Omega(p)^n$.
  We want to connect points in G to p.

The ACF formula

We use Alt-Caffarelli-Friedman (ACF) monotonicity formula to connect a corkscrew point $x \in \Omega$ to another point $x' \in \Omega$, with $\delta_\Omega(x') \approx 100\,\delta_\Omega(x)$.

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Theorem (ACF)

Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative subharmonic functions. Suppose that that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$. Set

$$J(x,r) = \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}} dy \right) \cdot \left(\frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}} dy \right).$$

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Then $J(x,\cdot)$ is non-decreasing in $r \in (0,R]$.

This formula is a basic tool in free boundary problems.
It can be used to "prove connectivity".

How to use the ACF formula

We want to connect $x, x' \in \Omega$. We know that

$$g(p,x) > \lambda \approx \frac{\delta_\Omega(x)}{\delta_\Omega(p)^n},$$

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Consider the functions

$$u_1(y) = (g(p,y) - \tfrac{1}{2}\lambda)^+ \, \chi_{U_x},$$

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If $u_1 \cdot u_2 \equiv 0$ we get a contradiction using the ACF formula.
In this way we can build "short paths".

Problem:
When we iterate many times the constants worsen and this collapses.

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Using a corona decomposition we combine the construction of short paths using ACF with geometric arguments.

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(a) *The family of roots of $\mathcal{T} \in I$ fulfils the packing condition*

$$\sum_{\mathcal{T} \in I : \mathrm{Root}(\mathcal{T}) \subset S} \mu(\mathrm{Root}(\mathcal{T})) \leq C\,\mu(S) \quad \text{for all } S \in \mathcal{D}_\mu.$$

(b) *In each $\mathcal{T} \in I$, E is "very well approximated" by an n-dimensional Lipschitz graph associated with $\mathcal{T}$ (using β coefficients, say).*

Thank you!