Bilinear Fourier integral operators with non-separable phase functions

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$$= \int_0^\infty m(x,t,u,v)T^{\phi_1}(Q_t^u(f))(x)T^{\phi_2}(P_t^v(g))(x)\frac{dt}{t}.$$

It now compares well to the paraproduct $\int_0^\infty Q_t(f)(x)P_t(g)(x)\frac{dt}{t}$

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Examples from PDEs point to studying

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- Concerning the 'end-point' cases, linear $H^1 \to L^1$ boundedness is no longer good enough.

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Can we removing the compact x-support? [Ruzhansky-Sugimoto '15] provided a method to go from local boundedness to global boundedness

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Can we removing the compact x-support? [Ruzhansky-Sugimoto '15] provided a method to go from local boundedness to global boundedness and applied it in the setting of L^p spaces.

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and again,

$$\begin{aligned}
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&\le \sum_{2^j \ge r^{-1}} \underbrace{\int_{(B^*)^c} |T_j(f)(x)|^p dx}_{\substack{\uparrow \\ \text{(iii)}}} + \sum_{2^j < r^{-1}} \underbrace{\int_{(B^*)^c} |T_j(f)(x)|^p dx}_{\substack{\uparrow \\ \text{(ii)}}}
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High-frequency part, small balls

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&\leq \sum_{2^j \geq r^{-1}} A 2^{-j} r^{-1} 2^{jn(p-1)} \left| \int_B f(y) dy \right|^p
\end{aligned}
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High-frequency part, small balls

$$\sum_{2^j \geq r^{-1}} \int_{(B^*)^c} |T_j(f)(x)|^p dx$$

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