H^∞-calculus for the Stokes operator on bounded Lipschitz domains

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Outline of the talk

1. The Stokes operator in bounded Lipschitz domains
2. Result on H^∞-calculus
3. The proof
   (a) Conditions on Littlewood-Paley operators
   (b) Verifying assumptions in L^q
   (c) Verifying assumptions in L^2
4. Applications

1. The Stokes operator in bounded Lipschitz domains

Navier-Stokes equations on $\Omega \subseteq \mathbb{R}^d$ bounded Lipschitz

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} u &= 0, \quad \text{in } (0,T) \times \Omega, \\ u &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) &= u_0, \quad \text{in } \Omega. \end{aligned}$$

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Apply Helmholtz projection $\mathbb{P}$ onto divergence-free vector fields

$$\begin{aligned} \partial_t u - \mathbb{P}\Delta u + \mathbb{P}(u \cdot \nabla) u &= 0, \quad \text{in } (0,T) \times \Omega, \\ u &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) &= u_0 = \mathbb{P} u_0, \quad \text{in } \Omega. \end{aligned}$$

Stokes operator A_2 in L^2

For $q \in (1,\infty)$ let

$$\begin{aligned} L^q_\sigma(\Omega) &:= \overline{\{\varphi \in C_c^\infty(\Omega)^d : \mathrm{div}\,\varphi = 0\}}^{\|\cdot\|_q} \\ &= \{u \in L^q(\Omega)^d : \mathrm{div}\, u = 0, \nu \cdot u = 0 \text{ on } \partial\Omega\} \end{aligned}$$

The Stokes operator A_2 in $L^2_\sigma(\Omega)$ is associated with the form

$$\mathfrak{a}(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx$$

defined on

$$H^1_{0,\sigma}(\Omega) := \overline{\{\varphi \in C_c^\infty(\Omega)^d : \mathrm{div}\,\varphi = 0\}}^{\|\cdot\|_{H^1}} = H^1_0(\Omega)^d \cap L^2_\sigma(\Omega).$$

Compare: $-\Delta_2^D$ in $L^2(\Omega)^d$ is associated with $\mathfrak{a}$ defined on $H^1_0(\Omega)^d$.

Stokes operator in L^2 (cont.)

Then

- $A_2 \geq 0$ is self adjoint in $L^2_\sigma(\Omega)$,
- e^{-zA_2} is an exponentially stable analytic semigroup on $\{\operatorname{Re} z > 0\}$,
- A_2 has a functional calculus for bounded Borel functions on $[0, \infty)$,
- $D(A_2^{1/2}) = H^1_{0,\sigma}(\Omega)$,

Stokes operator in L^2 (cont.)

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- e^{-zA_2} is an exponentially stable analytic semigroup on $\{\mathrm{Re}\, z > 0\}$,
- A_2 has a functional calculus for bounded Borel functions on $[0, \infty)$,
- $D(A_2^{1/2}) = H^1_{0,\sigma}(\Omega)$,
- $D(A_2^{s/2}) = H^s_\sigma(\Omega) := H^s(\Omega)^d \cap L^2_\sigma(\Omega)$ for $s \in (0, 1/2)$
  $\rightarrow$ Mitrea Monniaux 2008.

Similar for $-\Delta_2^D$ in $L^2(\Omega)^d$. In particular

- $D((-\Delta_2^D)^{s/2}) = H^s(\Omega)^d$ for $s \in (0, 1/2)$.

Stokes operator in L^2 (cont.)

Question: Is $A_2 = -\mathbb{P}_2 \Delta_2^D$ with $D(A_2) = D(\Delta_2^D) \cap L^2_\sigma(\Omega)$?
Not quite.
However, for $\widetilde{A_2} : H^1_{0,\sigma}(\Omega) \to (H^1_{0,\sigma}(\Omega))^*$ we have (Monniaux 2006)

$$\widetilde{A_2} = -\widetilde{P_2}\widetilde{\Delta_2^D} J_2, \qquad \text{where}$$

$$\begin{aligned} J_2 : H^1_{0,\sigma}(\Omega) &\to H^1_0(\Omega)^d \quad \text{inclusion}, \\ \widetilde{P_2} = J_2^* : \left(H^1_0(\Omega)^d\right)^* &\to \left(H^1_{0,\sigma}(\Omega)\right)^* \quad \text{restriction}. \end{aligned}$$

Then

$$D(A_2) = \{u \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega) : \exists \phi \in L^2(\Omega) : \underbrace{-\Delta u + \nabla \phi}_{=A_2 u} \in L^2_\sigma(\Omega)\}.$$

Helmholtz decomposition in L^q

For $q = 2$ we have the *Helmholtz* (or *Leray*) *decomposition*

$$L^2(\Omega)^d = L^2_\sigma(\Omega) \oplus \nabla H^1(\Omega) \quad \text{(orthogonal)}$$

with *Helmholtz projection* $\mathbb{P}_2 : L^2(\Omega)^d \to L^2_\sigma(\Omega)$.

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For $q \neq 2$

$$L^q(\Omega)^d = L^q_\sigma(\Omega) \oplus \nabla W^{1,q}(\Omega)$$

holds if

- $d \geq 3$ and $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ (Fabes Mendez Mitrea 1998)
- $d = 2$ and $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$ (D. Mitrea 2002)

where $\varepsilon = \varepsilon(\Omega)$ and $\varepsilon = \infty$ for $\partial\Omega \in C^1$.

Stokes operator in L^q: Shen's result

Theorem (Shen 2012)

For any $\theta \in (\pi/2, \pi)$ there exists $\varepsilon > 0$, only depending on d, θ and the Lipschitz character of Ω, such that for

$$\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon$$

there is a constant $C_{q,\theta}$ satisfying

$$\|(\lambda + A_q)^{-1} f\|_{L^q} \leq \frac{C_{q,\theta}}{|\lambda| + 1} \|f\|_{L^q}, \quad \lambda \in \Sigma_\theta, f \in L^q_\sigma(\Omega),$$

where $C_{q,\theta}$ only depends on d, q, θ and Lipschitz character of Ω.

2. Result on H^∞-calculus

Theorem 1 (K. Weis 2017)

Let $\Omega \subseteq \mathbb{R}^d$ where $d \geq 3$ be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that, for

$$\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon,$$

the Stokes operator A_q has a bounded H^∞-calculus in $L^q_\sigma(\Omega)$.
For these q we have $D(A_q^\alpha) = D((-\Delta_q^D)^\alpha) \cap L^q_\sigma(\Omega)$ for

$$|\alpha| < \frac{1}{2} - \left(\frac{1}{d} + 2\varepsilon\right)^{-1}\left|\frac{1}{2} - \frac{1}{q}\right|.$$

3. The proof: (a) Conditions on Littlewood-Paley operators

H^∞-calculus characterized by Littlewood-Paley estimates like

$$\|x\|_{L^q} \sim \Big\| \Big(\sum_{n\in\mathbb{Z}} |\varphi(2^n A)x|^2 \Big)^{1/2} \Big\|_{L^q}, \quad x \in L^q(U),$$

McIntosh 1986; CDMY 1996; Kalton Weis 2001

Optimal angle is the angle of (almost) R-sectoriality,

McIntosh 1986, Kalton Weis 2001, Kalton K. Weis 2006.

Idea: Transfer H^∞-calculus from an operator B to an (almost) R-sectorial operator A via Littlewood-Paley square functions

Kalton K. Weis 2006.

Conditions on Littlewood-Paley operators: Result

Theorem 2 (K. Weis 2017)

Let B have a bounded $H^\infty(\Sigma_\sigma)$-calculus on a Banach space X and let A be an (almost) R-sectorial operator in X. Assume that there are functions $\varphi, \psi \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ where $\nu > \sigma$ such that, for some $\beta > 0$ and all $k \in \mathbb{Z}$,

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+k}A)\psi(t2^j B) : j \in \mathbb{Z}\} \le C2^{-\beta|k|}, \quad (1)$$

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+k}A)'\psi(t2^j B)' : j \in \mathbb{Z}\} \le C2^{-\beta|k|}. \quad (2)$$

Then A has a bounded H^∞-calculus on X.

Conditions on Littlewood-Paley operators: Basic idea for $X = L^p$

Assume $\sum_{n\in\mathbb{Z}} \psi^2(2^n\lambda) = 1$ for all λ. Then

$$
\begin{aligned}
& \Big\| \big(\sum_n |\varphi(2^n A)x|^2\big)^{1/2} \Big\|_{L^p} \\
= \quad & \Big\| \Big(\sum_n \big|\varphi(2^n A)\big[\sum_m \psi^2(2^{n+m}B)x\big]\big|^2\Big)^{1/2} \Big\|_{L^p} \\
\leq \quad & \sum_m \Big\| \Big(\sum_n |\varphi(2^n A)\psi(2^{n+m}B)\psi(2^{n+m}B)x|^2 \Big)^{1/2} \Big\|_{L^p}
\end{aligned}
$$

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= \; & \sum_m \Big\| \Big(\sum_n |\varphi(2^{n-m}A)\psi(2^n B)\psi(2^n B)x|^2\Big)^{1/2} \Big\|_{L^p} \\
\le \; & C\Big(\sum_m 2^{-\beta|m|}\Big) \Big\| \Big(\sum_n |\psi(2^n B)x|^2\Big)^{1/2} \Big\|_{L^p} \lesssim C' \|x\|_{L^p}.
\end{aligned}
$$

□

Conditions on Littlewood-Paley operators: Complemented subspaces

Theorem 2' (K. Weis 2017)

Let X and Y be Banach spaces. Let $R : Y \to X$ and $S : X \to Y$ satisfy $RS = I_X$. Let B have a bounded $H^\infty(\Sigma_\sigma)$-calculus in Y and let A be (almost) R-sectorial in X. Assume that there are functions $\varphi, \psi \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ where $\nu > \sigma$ such that for some $\beta > 0$

$$\sup_{1 \le s,t \le 2} \mathcal{R}\{\varphi(s2^{j+k}A) R \psi(t2^j B) : j \in \mathbb{Z}\} \le C2^{-\beta|k|} \quad (3)$$

$$\sup_{1 \le s,t \le 2} \mathcal{R}\{\varphi(s2^{j+k}A)' S' \psi(t2^j B)' : j \in \mathbb{Z}\} \le C2^{-\beta|k|}. \quad (4)$$

Then A has a bounded H^∞-calculus on X.

Conditions on Littlewood-Paley operators: How to get (3)?

Condition (3) holds if

$$R(D(B^\alpha)) \subseteq D(A^\alpha), \qquad \|A^\alpha R y\|_X \leq C\|B^\alpha y\|_Y, \tag{5}$$

for $\alpha = \alpha_1, \alpha_2$ where $\alpha_1 < 0 < \alpha_2$.

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for $\alpha = \alpha_1, \alpha_2$ where $\alpha_1 < 0 < \alpha_2$.
Proof:

$$\begin{aligned} & \varphi(t2^{j+k}A)R\psi(s2^j B) \\ = & \left(\frac{t}{s}\right)^\alpha 2^{\alpha k}(t2^{j+k})^{-\alpha}A^{-\alpha}\varphi(t2^{j+k}A)[A^\alpha R B^{-\alpha}](s2^j)^\alpha B^\alpha \psi(s2^j B) \\ = & \left(\frac{t}{s}\right)^\alpha 2^{\alpha k}\,\widetilde{\varphi}(t2^{j+k}A)\, M\, \widetilde{\psi}(s2^j B), \end{aligned}$$

where $\widetilde{\varphi}(z) = z^{-\alpha}\varphi(z)$, $\widetilde{\psi}(z) = z^\alpha \psi(z)$ are in H_0^∞ □

How to get (3)? (cont.)

Recall

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+k}A)R\psi(t2^jB) : j \in \mathbb{Z}\} \le C2^{-\beta|k|}. \qquad (3)$$

Now let $X = L^q_\sigma(\Omega)$, $Y = L^q(\Omega)^d$, $R = \mathbb{P}$. Assume

- condition (3) holds for $q = q_0$ and $\beta = 0$ (automatic),
- condition (3) holds for $q = 2$ and $\beta > 0$ (can use (5) for that).

By interpolation, (3) holds for q between 2 and q_0 and some $\beta > 0$.

How to get (3)? (cont.)

Recall

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+k}A)R\psi(t2^jB) : j\in\mathbb{Z}\} \le C2^{-\beta|k|}. \qquad (3)$$

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By interpolation, (3) holds for q between 2 and q_0 and some $\beta > 0$.

For the proof of Theorem 1 we thus need

- A_q is R-sectorial in $L^q_\sigma(\Omega)$ for the stated range of q,
- $\mathbb{P} : H^s(\Omega)^d \to H^s_\sigma(\Omega)$ and projection in $H^s(\Omega)^d$ for $|s| > 0$ small.

The proof: (b) Verifying assumptions in L^q

Proposition 3 (K. Weis 2017, Tolksdorf 2017)

For any $\theta \in (\pi/2, \pi)$ there exists $\varepsilon > 0$, only depending on d, θ and the Lipschitz character of Ω, such that for

$$\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon$$

there exists a constant $\tilde{C}_{q,\theta}$, only depending on d, q, θ and Lipschitz character of Ω, such that

$$\mathcal{R}\{(|\lambda| + 1)(\lambda + A_q)^{-1} : \lambda \in \Sigma_\theta\} \leq \tilde{C}_{q,\theta},$$

where the R-bound is taken for operators $L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$.

Sketch of proof (Proposition)

Proceed as in Shen 2012:

- $q > 2$ suffices (by self duality)
- use Shen's extrapolation lemma:
  needs estimate $L^2 \to L^q$ for the resolvent problem
- pass through L^2-estimates for boundary value problems
- $L^2 \to L^q$-estimate just needed for a single operator

Extend this to square functions!

□

The proof: (c) Verifying assumptions in L^2

Proposition (Mitrea Monniaux 2008)

For $|s| < \frac{1}{2}$, the Helmholtz projection $\mathbb{P}$ acts as a bounded linear projection P_s in $H^s(\Omega)^d$ and yields the decomposition

$$H^s(\Omega)^d = H^s_\sigma(\Omega) \oplus \nabla H^{s+1}(\Omega)$$

as a topological direct sum.

The proof uses arguments from Fabes Mendez Mitrea 1998 and

$$\mathbb{P}u = u - \nabla \mathrm{div}\, \Pi_\Omega(u) - \nabla\psi,$$

where Π_Ω is the Newton potential and ψ solves

$$\Delta\psi = 0, \qquad \frac{\partial\psi}{\partial\nu} = \nu \cdot (u - \nabla \mathrm{div}\, \Pi_\Omega(u)).$$

4. Applications

Theorem (Tolksdorf 2018)

Let $\Omega \subseteq \mathbb{R}^d$ where $d \geq 3$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for

$$\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon,$$

the Stokes operator A_q satisfies

$$D(A_q^{1/2}) = W^{1,q}_{0,\sigma}(\Omega).$$

$\rightarrow$ L^p-L^q-mapping properties and gradient estimates for semigroup

$\rightarrow$ regularity results for Navier-Stokes equations complementing those by Mitrea, Monniaux 2008.