Recent approaches based on harmonic analysis for the study of non regular solutions to the Navier-Stokes equations with variable density

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The classical incompressible Navier-Stokes equations:

$$(NS): \begin{cases} u_t + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases}$$

Here $u = u(t,x) \in \mathbb{R}^d$ and $P = P(t,x) \in \mathbb{R}$ with $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^d$, $d \geq 2$.

- Energy balance: $\frac{1}{2}\|u(t)\|^2_{L^2} + \mu \int_0^t \|\nabla u\|^2_{L^2}\, d\tau = \frac{1}{2}\|u_0\|^2_{L^2}.$
- Scaling invariance: If $\Omega = \mathbb{R}^d$ then the System (NS) is invariant (up to a change of P and u_0) by the family of dilations:

$$T_\lambda u(t,x) := \lambda u(\lambda^2 t, \lambda x).$$

Global weak solutions

The classical incompressible Navier-Stokes equations:

$$(NS):\begin{cases} u_t+\operatorname{div}(u\otimes u)-\mu\Delta u+\nabla P=0 & \text{in } \mathbb{R}_+\times\Omega\\ \operatorname{div}u=0 & \text{in } \mathbb{R}_+\times\Omega\\ u=0 & \text{on } \mathbb{R}_+\times\partial\Omega\\ u|_{t=0}=u_0 & \text{in } \Omega.\end{cases}$$

Theorem (J. Leray, 1934)

Any divergence free $u_0\in L^2(\Omega)$ *generates at least one global weak solution of* (NS) *satisfying the energy inequality:*

$$\frac12\|u(t)\|_{L^2}^2+\mu\int_0^t\|\nabla u\|_{L^2}^2\,d\tau\leq\frac12\|u_0\|_{L^2}^2.$$

- The proof relies essentially on the energy balance and on compactness arguments (or, equivalently, Schauder-Tikhonov theorem).
- Unless $d=2$, uniqueness of Leray's solutions is (still) an open question.

'Mild solutions' of NS equations

Let $A = -\mu\Delta u + \nabla P$ be the Stokes operator. Then, formally,

$$u(t) = \underbrace{e^{-tA}u_0}_{u_L} - \underbrace{\int_0^t e^{-(t-\tau)A}(\mathrm{div}\,(u\otimes u)(\tau))\,d\tau}_{\mathcal{B}(u,u)}.$$

Lemma (based on the fixed point theorem in a Banach spaces)

Let X be a Banach space and $\mathcal{B}: X\times X \to X$, a continuous bilinear map with norm M. Then equation $\boxed{u = u_L - \mathcal{B}(u,u)}$ *has a unique solution in the closed ball $\overline{B}(0, 2\|u_L\|_X)$ whenever*

$$4M\|u_L\|_X < 1.$$

- The largest spaces in which one may expect $\mathcal{B}$ to be continuous are *scaling invariant* by the family of dilations $(T_\lambda)_{\lambda>0}$.

- Examples : small initial data in Sobolev spaces $\dot H^{\frac d2-1}(\mathbb{R}^d)$ (Fujita-Kato), Lebesgue space $L^d(\mathbb{R}^d)$ (Giga- Kato), Besov spaces $\dot B^{\frac dp-1}_{p,r}(\mathbb{R}^d)$, etc.

The inhomogeneous **incompressible Navier-Stokes equations** read:

$$(INS):\begin{cases}(\rho u)_t+\operatorname{div}(\rho u\otimes u)-\mu\Delta u+\nabla P=0 & \text{in } \mathbb{R}_+\times\Omega\\ \operatorname{div} u=0 & \text{in } \mathbb{R}_+\times\Omega\\ \rho_t+\operatorname{div}(\rho u)=0 & \text{in } \mathbb{R}_+\times\Omega.\end{cases}$$

- Energy balance : $\frac{1}{2}\|\sqrt{\rho(t)}\,u(t)\|_{L^2}^2+\mu\int_0^t\|\nabla u\|_{L^2}^2\,d\tau=\frac{1}{2}\|\sqrt{\rho_0}\,u_0\|_{L^2}^2.$
- Conservation of L^p norms of functions of the density.
- Scaling invariance if $\Omega=\mathbb{R}^d$:

$$\rho(t,x)\to\rho(\lambda^2t,\lambda x),\qquad u(t,x)\to\lambda u(\lambda^2t,\lambda x),\qquad P(t,x)\to\lambda^2P(\lambda^2t,\lambda x).$$

- **Global weak solutions with finite energy** for any pair (ρ_0,u_0) such that $\rho_0\in L^\infty(\Omega)$ with $\rho_0\geq 0$, and $\sqrt{\rho_0}u_0\in L^2(\Omega)$ with $\operatorname{div} u_0=0$ (Kazhikhov, 1974, J. Simon 1990, P.-L. Lions 1996).
- Even if $d=2$, uniqueness in the class of finite energy solutions is a widely open question.
- **Strong solutions for smooth data** (global if $d=2$ and $\inf\rho_0>0$) : Ladyzhenskaya and Solonnikov (1978).

Is (INS) a good model for mixture of non-reacting fluids ?

1. Can we solve uniquely (INS) if ρ_0 is *discontinuous* across an interface ?
2. Is the solution unique for such a ρ_0 ?
3. Can we allow for vacuum regions ?
4. Is the regularity of interfaces preserved during the evolution ?

• We expect the interfaces to be transported by the flow of u. Hence, by Cauchy-Lipschitz theorem, the minimal requirement for preserving their regularity is $\nabla u \in L^1(0,T;L^\infty(\Omega))$.
It will be also needed for uniqueness.

• Even for $d=2$ and for the heat equation, having just $u_0 \in L^2(\Omega)$ does not ensure $\nabla u \in L^1(0,T;L^\infty(\Omega))$.

Aim of the talk

Presenting three different (and complementary) approaches:

1. Critical functional framework and endpoint maximal regularity;
2. Classical maximal regularity;
3. Energy approach.

I. An approach based on the endpoint maximal regularity

For simplicity, we assume that $\Omega = \mathbb{R}^d$ $(d \geq 2)$ and that $\rho \to 1$ at ∞.

Set $a := \rho - 1$. System for (a, u, P) reads:

$$(INS) : \begin{cases} u_t - \mu\Delta u + \nabla P = -au_t - (1+a)\mathrm{div}\,(u \otimes u) & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ \mathrm{div}\, u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ a_t + u \cdot \nabla a = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Scaling invariance:

$$a(t,x) \to a(\lambda^2 t, \lambda x), \qquad u(t,x) \to \lambda u(\lambda^2 t, \lambda x), \qquad P(t,x) \to \lambda^2 P(\lambda^2 t, \lambda x).$$

- *Endpoint maximal regularity* for the Stokes system:

$$(S):\begin{cases} u_t-\mu\Delta u+\nabla P=f & \text{in }\ \mathbb{R}_+\times\mathbb{R}^d\\ \operatorname{div} u=0 & \text{in }\ \mathbb{R}_+\times\mathbb{R}^d.\end{cases}$$

We have for any $s\in\mathbb{R}$ and $p\in[1,\infty]$:

$$\|u\|_{L^\infty(\mathbb{R}_+;\dot B^s_{p,1})}+\|u_t,\mu\nabla_x^2u,\nabla_xP\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1})}\lesssim\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1})}.$$

Scaling invariance pushes us to take $s=\frac{d}{p}-1$, and thus $(u,\nabla P)\in E_p$ with

$$E_p=\big\{(u,\nabla P)\in\mathcal{C}_b(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})\times L^1(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})\ \text{ with }\ u_t,\nabla^2u\in L^1(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})\big\}.$$

- *Stability of the Besov space* $\dot B^{\frac dp}_{p,1}$ by product if $p<\infty$:

$$\|\operatorname{div}(u\otimes u)\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\|u\otimes u\|_{\dot B^{\frac dp}_{p,1}}\lesssim\|u\|^2_{\dot B^{\frac dp}_{p,1}}.$$

- *Multiplier spaces:* $\|a\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}:=\sup_{\|z\|_{\dot B^{\frac dp-1}_{p,1}}=1}\|az\|_{\dot B^{\frac dp-1}_{p,1}}<\infty.$

- *Estimates for the transport equation* (deduced from the ones in Besov spaces):

$$\|a(t)\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}\leq\|a_0\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}\exp\Big\{C\int_0^t\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}\,d\tau\Big\}.$$

Taking $f=-au_t-(1+a)\mathrm{div}\,(u\otimes u)$ in (S), we deduce that

$$\|(u,\nabla P)\|_{E_p}\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\|u_t\|_{L^1(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})}+\Big(1+\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\Big)\|u\|^2_{E_p}.$$

Combining with

$$\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\leq\|a_0\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}\exp\Big\{C\|(u,\nabla P)\|_{E_p}\Big\},$$

one may close the estimates *if both* $\|a_0\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}$ *and* $\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}$ *are small.*

Theorem (D & P.B. Mucha, 2012)

Assume that $1\leq p<2d$. *There exists a constant* $c>0$ *such that if*

$$\mu\|a_0\|_{\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})\cap L^\infty}+\|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\leq c\mu \qquad (1)$$

then (INS) *has a unique solution with* $(u,\nabla P)\in E_p$ *and* $a\in\mathcal{C}(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1}))$.

The direct proof: NO CONTRACTING MAPPING ARGUMENT.

1. Constructing a sequence of approximate solutions and uniform estimates;
2. Compactness;
3. Uniqueness : loss of one derivative. PROBLEM HERE.

Corollary (The density patch problem)

Let D *be a* C^1 *bounded domain. If* u_0 *fulfills* (1) *with* $d-1<p<2d$ *and* $\rho_0=c_1 1_D+c_2 1_{^cD}$ *with* $|c_1-c_2|\ll 1$ *then* (INS) *has a unique global solution as above, and* $\rho(t)=c_1 1_{D_t}+c_2 1_{^cD_t}$. *Furthermore* D_t **remains** C^1.

Lagrangian coordinates: Assume $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty)$ and set

$$\bar\rho(t,y) := \rho(t,x), \quad \bar u(t,y) := u(t,x) \ \text{ and } \ \bar P(t,y) := P(t,x) \ \text{ with } \ \boxed{x := X(t,y)}$$

where X **is the flow of** u defined by

$$\boxed{X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau.}$$

(INS) in Lagrangian coordinates:

- $\bar\rho$ is time independent.
- $(\bar u, \bar P)$ satisfies

$$(\widetilde{INS}) : \begin{cases} \rho_0 \bar u_t - \mathrm{div}\,(A^T A \nabla \bar u) + {}^T A \cdot \nabla \bar P = 0, \\ \mathrm{div}\,(A \bar u) = {}^T A : \nabla \bar u = 0, \end{cases}$$

with $A = (D_y X)^{-1} = \displaystyle\sum_{k=0}^{+\infty} (-1)^k \left(\int_0^t D\bar u(\tau,\cdot)\,d\tau \right)^k.$

- $(\widetilde{INS})$ may be solved *by means of the fixed point theorem.*
- Uniqueness may be proved at the level of Lagrangian coordinates.

II. An approach based on the classical maximal regularity

Consider a solution $(u, \nabla P)$ to

$$(S): \begin{cases} u_t - \mu\Delta u + \nabla P = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Then, for all $1 < p, r < \infty$,

$$|(u, \nabla P)\|_{E_p^r} := \|(u_t, \mu\nabla^2 u, \nabla P)\|_{L^r(\mathbb{R}_+; L^p)} + \|u\|_{L^\infty(\mathbb{R}_+; \dot{B}^{2-\frac{2}{r}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^{2-\frac{2}{r}}_{p,r}} + \|f\|_{L^r(\mathbb{R}_+; L^p)}.$$

- *Critical regularity* for (INS) corresponds to

$$2 - \frac{2}{r} = \frac{d}{p} - 1.$$

which gives us the constraint $\frac{d}{3} < p < d.$

- We want to apply this to $f = -au_t - (1 + a)u \cdot \nabla u.$

So we have

$$\|(u,\nabla P)\|_{E_p^r} := \|(u_t,\mu\nabla^2 u,\nabla P)\|_{L^r(\mathbb{R}_+;L^p)} + \|u\|_{L^\infty(\mathbb{R}_+;\dot B^{2-\frac2r}_{p,r})} \lesssim \|u_0\|_{\dot B^{2-\frac2r}_{p,r}}$$
$$+\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)}\|u_t\|_{L^r(\mathbb{R}_+;L^p)} + (1+\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)})\|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)}.$$

Note that $\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)} = \|a_0\|_{L^\infty}$. Hence, if $\|a_0\|_{L^\infty}$ is small, then we get

$$\|(u,\nabla P)\|_{E_p^r} \lesssim \|u_0\|_{\dot B^{2-\frac2r}_{p,r}} + \|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)}.$$

If *critical regularity*: $2-\frac2r = \frac dp - 1$ then we have

$$\|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)} \le \|u\|_{L^{2r}(\mathbb{R}_+;L^{\frac{dr}{r-1}})}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L^{\frac{dr}{2r-1}})}$$

and

$$\|u\|_{L^{\frac{dr}{r-1}}} \lesssim \|\nabla u\|_{L^{\frac{dr}{2r-1}}} \qquad \text{(Sobolev embedding)}$$
$$\|\nabla u\|_{L^{\frac{dr}{2r-1}}} \lesssim \|\nabla^2 u\|_{L^p}^{\frac12}\|u\|^{\frac12}_{\dot B^{2-\frac2r}_{p,r}} \qquad \text{(Interpolation).}$$

Hence

$$\|(u,\nabla P)\|_{E_p^r} \lesssim \|u_0\|_{\dot B^{2-\frac2r}_{p,r}} + \|(u,\nabla P)\|^2_{E_p^r}.$$

Results

Theorem (Huang, Paicu & Zhang, 2013)

Let $a_0 \in L^\infty(\mathbb{R}^d)$ *and* $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d)$ *with* $d \geq 2,$ $p := \frac{dr}{3r-2}$ *and* $r \in (1,\infty).$ *There exists a positive constant* $c_0 = c_0(r,d)$ *so that if*

$$\mu\|a_0\|_{L^\infty} + \|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}} \leq c_0\mu \tag{2}$$

then (NSI) *has a global solution* $(a,u,\nabla P)$ *satisfying* $\|a(t)\|_{L^\infty} = \|a_0\|_{L^\infty}$ *for all* $t \geq 0,$ *and* $(u,\nabla P) \in E^r_p.$

Since $r > 1$ and $p < d$, we do not have $\nabla u \in L^1_{loc}(\mathbb{R}_+;L^\infty)$ which precludes our using Lagrangian coordinates for proving uniqueness.

Theorem (Huang, Paicu & Zhang, 2013)

If, in addition, $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}$ *for some* $d < \widetilde{p} \leq \frac{dr}{r-1},$ *then* $(u,\nabla P)$ *also belongs to* $E^r_{\widetilde{p}},$ *and the solution* $(a,u,\nabla P)$ *is unique in the space* $L^\infty(\mathbb{R}_+\times\mathbb{R}^d)\times\big(E^r_p\cap E^r_{\widetilde{p}}\big).$

Comments on the first two approaches

- *Adaptable to more general domains* : half-space, bounded smooth domain and exterior smooth domain (more tricky);

- *Local in time results* are provable, but, still, a_0 has to be small for some suitable norm since our approach relies on the Stokes system with constant coefficients.

- We get nothing special for $d = 2$ even though global existence is expected with no smallness condition at all.

- The *viscosity* coefficient has to be *independent of the density.*

An approach based on energy estimates

- The basic energy balance:

$$\frac{1}{2}\|\sqrt{\rho(t)}\,u(t)\|_{L^2}^2+\mu\int_0^t\|\nabla u\|_{L^2}^2\,d\tau=\frac{1}{2}\|\sqrt{\rho_0}\,u_0\|_{L^2}^2.$$

- All L^p norms of the functions of the density are preserved through the evolution.

However, even if $d=2,$ those relations are far from being enough to ensure uniqueness and conservation of geometric structures like interfaces between different densities.

In dimension 2, critical regularity is $u_0\in L^2$. What if one starts with $u_0\in H^1$?

Theorem (D & P.B. Mucha, 2017)

Consider **any data** (ρ_0, u_0) *in* $L^\infty(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ *with* $\rho_0 \geq 0$ *and* $\operatorname{div} u_0 = 0$.

Then System (INS) *supplemented with* (ρ_0, u_0) *admits a* **unique global solution** $(\rho, u, \nabla P)$ *that satisfies the energy equality, the conservation of total mass and momentum,*

$$\rho \in L^\infty(\mathbb{R}_+; L^\infty), \quad u \in L^\infty(\mathbb{R}_+; H^1), \quad \sqrt{\rho} u_t, \nabla^2 u, \nabla P \in L^2(\mathbb{R}_+; L^2)$$

and also, for all $1 \leq r < 2$, $1 \leq m < \infty$ *and* $T \geq 0$,

$$\nabla(\sqrt{t} P),\ \nabla^2(\sqrt{t} u) \in L^\infty(0, T; L^r) \cap L^2(0, T; L^m).$$

Furthermore, we have $\sqrt{\rho} u \in \mathcal{C}(\mathbb{R}_+; L^2)$ *and* $\rho \in \mathcal{C}(\mathbb{R}_+; L^p)$ *for all* $p < \infty$.

Corollary (The density patch problem)

If $u_0 \in H^1(\mathbb{T}^2)$ *and* $\rho_0 = c_1 1_D + c_2 1_{^cD}$ **with** $c_1, c_2 \geq 0$ **arbitrary** *and* D *a* $C^{1,\alpha}$ *open set with* $\alpha < 1$, *then* (INS) *has a unique global solution as above, and* $\rho(t) = c_1 1_{D_t} + c_2 1_{^cD_t}$. *Furthermore* D_t **remains** $C^{1,\alpha}$.

Main steps of the proof:

1. Global-in-time estimates for Sobolev regularity of u.
2. Sobolev regularity of u_t and time weights.
3. Shift of regularity and integrability : from time to space variable.
4. The existence scheme.
5. *Lagrangian coordinates* and uniqueness.

Assume with no loss of generality that

$$\int_{\mathbb{T}^2} \rho_0 \, dx = \mu = 1 \quad \text{and} \quad \int_{\mathbb{T}^2} \rho_0 u_0 \, dx = 0.$$

Step 1: global-in-time Sobolev estimates

- Take the L^2 scalar product of $\rho(u_t + u\cdot\nabla u) - \Delta u + \nabla P = 0$ with u_t:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla u|^2dx + \int_{\mathbb{T}^2}\rho|u_t|^2\,dx \leq \frac{1}{2}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

Let $\rho^* := \|\rho\|_{L^\infty}$. As $\boxed{-\Delta u + \nabla P = -\rho u_t - \rho u\cdot\nabla u \;\text{ and }\; \operatorname{div}\Delta u = 0}$, we get

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 = \|\rho(\partial_t u + u\cdot\nabla u)\|_{L^2}^2 \leq 2\rho^*\left(\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx\right)\cdot$$

Hence

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \frac{1}{4\rho^*}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right) \leq \frac{3}{2}\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

- Apply Hölder and Gagliardo-Nirenberg inequality, and use $\rho \leq \rho^* := \|\rho_0\|_{L^\infty}$:

$$\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx \leq \rho^*\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2 \;\leq C\rho^*\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}$$
$$\leq \frac{1}{12\rho^*}\|\nabla^2 u\|_{L^2}^2 + C(\rho^*)^3\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

- If $\rho \geq \rho_* > 0$, then $\|u\|_{L^2}^2 \leq \rho_*^{-1}\|\sqrt{\rho}u\|_{L^2}^2$. Combine the basic energy inequality with Gronwall lemma.

Step 1: global-in-time Sobolev estimates (continued)

Lemma (B. Desjardins, 1997)

If $\int_{\mathbb{T}^2} \rho\,dx = 1$ *and* $\int_{\mathbb{T}^2} \rho z\,dx = 0$ *then*

$$\left(\int_{\mathbb{T}^2} \rho z^4\,dx\right)^{\frac12} \leq C\|\sqrt{\rho}z\|_{L^2}\|\nabla z\|_{L^2}\log^{\frac12}\left(e+\|\rho-1\|_{L^2}^2+\frac{\rho^*\|\nabla z\|_{L^2}^2}{\|\sqrt{\rho}z\|_{L^2}^2}\right). \qquad (3)$$

- Write $\displaystyle\int_{\mathbb{T}^2} \rho|u\cdot\nabla u|^2 dx \leq \sqrt{\rho^*}\left(\int_{\mathbb{T}^2}\rho|u|^4\,dx\right)^{\frac12}\|\nabla u\|_{L^4}^2$

and use (3) with $z=u$, energy balance and $ab\leq a^2/2+b^2/2$:

$$\int_{\mathbb{T}^2} \rho|u\cdot\nabla u|^2 dx \leq \frac{1}{12\rho^*}\|\nabla^2 u\|_{L^2}^2$$
$$+C(\rho^*)^2\|\sqrt{\rho_0}\,u_0\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2\log\left(e+\|\rho_0-1\|_{L^2}^2+\rho^*\frac{\|\nabla u\|_{L^2}^2}{\|\sqrt{\rho_0}\,u_0\|_{L^2}^2}\right).$$

Step 1: global-in-time Sobolev estimates (continued)

We eventually get

$$\frac{d}{dt}X \le fX \log(e+X),$$

with $f(t) := C_0\|\nabla u(t)\|_{L^2}^2$ for some suitable $C_0 = C(\rho_0, u_0)$ and

$$X(t) := \int_{\mathbb{T}^2} |\nabla u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\Big(\rho|u_t|^2 + \frac{1}{4\rho^*}\big(|\nabla^2 u|^2 + |\nabla P|^2\big)\Big)dx.$$

Hence

$$(e+X(t)) \le (e+X(0))^{\exp(\int_0^t f(\tau)\,d\tau)} \le (e+X(0))^{\exp\,(C_0\|\sqrt{\rho_0}u_0\|_{L^2}^2)}.$$

- However $X<\infty$ does not imply $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^2))$.

Step 2: Sobolev regularity of u_t

Take the L^2 scalar product of $\rho(u_t+u\cdot\nabla u)-\Delta u+\nabla P=0$ with u_{tt}:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx+\int_{\mathbb{T}^2}|\nabla u_t|^2\,dx=\int_{\mathbb{T}^2}(\rho_t u_t-\rho_t u\cdot\nabla u-\rho u_t\cdot\nabla u)\cdot u_t\,dx.$$

- For $(\sqrt{\rho}u_t)|_{t=0}$ to be defined, we need the compatibility condition

$$-\Delta u_0=\sqrt{\rho_0}g-\nabla P_0\quad\text{with}\quad g\in L^2. \tag{4}$$

- Condition (4) *is not* needed if *we compensate the singularity at time* 0 *by some power of* t: take the scalar product of $\rho(u_t+u\cdot\nabla u)-\Delta u+\nabla P=0$ with tu_{tt}:

$$\|\sqrt{\rho t}\,u_t\|_{L^2}+\int_0^t\|\nabla\sqrt{\tau}\,u_t\|_{L^2}^2\,d\tau\leq h(t),$$

where h is a nondecreasing nonnegative function with $h(0)=0$ (use Step 1 and energy identity).

Step 3: Shift of regularity from time to space variable

- Step 1 gives $\nabla u\in L^\infty(\mathbb{R}_+;L^2),\;\;\nabla u\in L^2(\mathbb{R}_+;H^1),\;\;\nabla P,\sqrt{\rho}u_t\in L^2(\mathbb{R}_+\times\mathbb{T}^2).$
- Step 2 gives $\sqrt{\rho t}\,u_t\in L^\infty_{loc}(\mathbb{R}_+;L^2)$ and $\nabla\sqrt{t}\,u_t\in L^2_{loc}(\mathbb{R}_+;L^2).$
- Step 3: Use Stokes equation:

$$\begin{cases} -\Delta\sqrt{t}\,u+\nabla\sqrt{t}\,P=-\sqrt{t}\,\rho u_t-\sqrt{t}\,\rho u\cdot\nabla u,\\ \operatorname{div}\sqrt{t}\,u=0.\end{cases}$$

  Steps 1,2 + embedding imply that for all $T>0,$ the right-hand side is almost in $L^2(0,T;L^\infty).$ Hence so do $\nabla^2\sqrt{t}\,u$ and $\nabla\sqrt{t}\,P.$

• This implies that $\nabla u\in L^1_{loc}(0,T;L^\infty)$ from which one may go to Lagrangian coordinates and prove uniqueness.