Gundy-Varopoulos martingale transforms and their projection operators on manifolds and vector bundles

Li CHEN
joint with R. Bañuelos (Purdue) and F. Baudoin (UCONN)

University of Connecticut

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Riesz transforms on $\mathbb{R}^d$

On $\mathbb{R}^d$, Riesz transforms R_j, $j = 1, \cdots, d$, are formally defined by

$$R_j = \partial_j(-\Delta)^{-1/2}.$$

The classical Calderón-Zygmund theory gives

$$\|R_j f\|_p \leq C_p \|f\|_p.$$

The sharp inequality can be obtained by either analytic or probabilistic approach.

Theorem (Iwaniec-Martin 1996, Bañuelos-Wang 1995)

$$\|R_j f\|_p \leq \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p, \quad \forall p > 1$$

where $p^* = \max\{p, \frac{p}{p-1}\}$.

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Riesz transforms in different geometric settings

The study of Riesz transforms in different geometric settings was introduced by Stein (1970) on Lie groups and by Strichartz (1983) on manifolds.

- **Singular integral techniques and heat kernel estimates** See for instance [Duong-McIntosh 1995], [Coulhon-Duong 1999], [Auscher-Coulhon-Duong-Hofmann 2004], etc.
- **Probabilistic methods**
  - Martingale approach via Littlewood-Paley inequalities ([Bakry 1987])
  - Martingale transforms, which allow us to have sharp or at least dimension free estimates!

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Gundy-Varopoulos representation on $\mathbb{R}^d$

In 1979, Gundy and Varopoulos proved the now classical representation of Riesz transforms using "background radiation" process:

$$R_j f = -2 \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(\int_0^\tau A_j(\nabla, \partial_y)^{\mathrm{T}} Qf(\beta_s, B_s)(d\beta_s, dB_s) \mid \beta_\tau = x \right),$$

where

- $A_j = (a_{ik})$ is a $(d+1) \times (d+1)$ matrix with $a_{(d+1)j} = 1$ and otherwise 0;
- β_t: Brownian motion on $\mathbb{R}^d$ with initial distribution dx;
- B_t: Brownian motion on $\mathbb{R}$ with generator $\frac{d^2}{dy^2}$ starting from $y_0 > 0$;
- $\tau = \inf\{t > 0 : B_t = 0\}$: the stopping time;
- $Qf(x,y) = e^{-y\sqrt{-\Delta}} f(x)$: the harmonic extension of $f \in C_0^\infty(\mathbb{R}^d)$.

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Sharp inequalities for martingales

$(\Omega, \mathcal{F}, \mathbb{P})$: a complete probability space;

X and Y: adapted, real-valued martingales which have right-continuous paths with left-limits (càdlàg);

Y is differentially subordinate to X: $|Y_0| \leq |X_0|$ and $\langle X \rangle_t - \langle Y \rangle_t$ is a nondecreasing and nonnegative function of t;

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Bañuelos and Wang proved the following sharp inequality extending the classical results of Burkholder (1966).

Theorem (Bañuelos-Wang 1995)

Let X and Y be two martingales with continuous paths such that Y is differentially subordinate to X. Fix $1 < p < \infty$, then

$$\|Y\|_p \leq (p^* - 1)\|X\|_p.$$

Furthermore, suppose the martingales X and Y are orthogonal. Then

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And afterwards

- Bañuelos-Wang 1995: sharp estimates for R_j;
- Arcozzi 1998: sharp estimates on Lie group of compact type;
- X.-D. Li 2008-2014: dimension free estimates on Riemmanian manifolds and on forms with curvature assumption;
- More recent:
  [Bañuelos-Baudoin 2013] on second order Riesz transforms on Lie groups of compact type,
  [Bañuelos-Osękowski 2015] on sharp martingale inequalities and Riesz transforms on manifolds, Lie groups and Gauss space,
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Scalar operators constructed from martingale transforms

$\mathbb{M}$: smooth manifold with smooth measure μ.

$X_1, \cdots, X_d$: locally Lipschitz vector fields on $\mathbb{M}$.

$V : \mathbb{M} \to \mathbb{R}$ is a non-positive smooth potential. Consider the Schrödinger operator

$$L = -\sum_{i=1}^{d} X_i^* X_i + V.$$

We can write

$$L = \sum_{i=1}^{d} X_i^2 + X_0 + V,$$

for some locally Lipchitz vector field X_0.

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Diffusion process

Let $(Y_t)_{t\geq 0}$ be the diffusion process on $\mathbb{M}$ with generator $\sum_{i=1}^{d} X_i^2 + X_0$ starting from the distribution μ.

Via Stratonovitch stochastic differential equation,

$$dY_t = X_0(t)dt + \sum_{i=1}^{d} X_i(Y_t) \circ d\beta_t^i,$$

where β_t is the Brownian motion on $\mathbb{R}^d$.

For $f \in C_0^\infty(\mathbb{M})$, denote

$$Q^V f(x,y) = P_y f(x) = e^{-y\sqrt{-L}} f(x).$$

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Projection operators

Consider the operators

$$T_i = \int_0^{+\infty} y P_y \left(\sqrt{-L} X_i - X_i^* \sqrt{-L} \right) P_y dy, \quad \forall 1 \leq i \leq d.$$

Theorem (Bañuelos-Baudoin-C. 2018)

For $f \in \mathcal{S}(\mathbb{M})$ *and* $1 \leq i \leq d$,

$$T_i f(x) = -\frac{1}{2} \lim_{y_0 \to \infty}$$

$$\mathbb{E}_{y_0} \left(e^{\int_0^\tau V(Y_v) dv} \int_0^\tau e^{-\int_0^s V(Y_v) dv} A_i(\nabla, \partial_y)^{\mathrm{T}} Q f(Y_s, B_s)(d\beta_s, dB_s) \mid Y_\tau = x \right)$$

where $\nabla = (X_1, \cdots, X_d)$, *and* A_i *is a* $(d+1) \times (d+1)$ *matrix with* $a_{i(d+1)} = -1$, $a_{(d+1)i} = 1$ *and otherwise* 0.

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Sharp estimate of Bañuelos and Osękowski

Theorem (Bañuelos-Osękowski 2015)

Let X and Y be two martingales with continuous paths such that Y is differentially subordinate to X. Consider the process

$$Z_t = e^{\int_0^t V_s ds} \int_0^t e^{-\int_0^s V_v dv} dY_s,$$

where $(V_t)_{t \geq 0}$ is a non-positive adapted and continuous process.

$$\|Z\|_p \leq (p^* - 1)\|X\|_p.$$

Main result

Theorem (Bañuelos-Baudoin-C. 2018)

Fix $1 < p < \infty$. *Then for every* $f \in \mathcal{S}(\mathbb{M})$,

$$\|T_i f\|_p \le \left(\frac{3}{2}\right)(p^* - 1)\|f\|_p.$$

If the potential $V \equiv 0$, *then*

$$\|T_i f\|_p \le \frac{1}{2}\cot\left(\frac{\pi}{2p^*}\right)\|f\|_p.$$

- **Applications:** Lie group of compact type, Heisenberg groups, $\mathbb{SU}(2)$, etc.

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Example: Lie group of compact type

G: Lie group of compact type with a bi-invariant Riemannian structure.

$X_1, \cdots, X_d$: an orthonormal basis of $\mathfrak{g}$.

$L = \sum_{i=1}^{d} X_i^2$: the Laplace-Beltrami operator.

We observe

$$T_i = \int_0^{+\infty} y P_y \left(\sqrt{-L} X_i - X_i^* \sqrt{-L} \right) P_y dy = \frac{1}{2} X_i (\sqrt{-L})^{-1}.$$

Proposition

$$\| X_i (\sqrt{-L})^{-1} \|_{L^p \to L^p} \leq \cot \left(\frac{\pi}{2p^*} \right).$$

This inequality was first proved by [Accozzi 1998].

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$$\mathbb{H}^n = \{(x,y,z) : x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

endowed with the group law

$$(x,y,z)\cdot(x',y',z') = \left(x+x', y+y', z+z' + \frac{1}{2}\left(\langle x, y'\rangle_{\mathbb{R}^n} - \langle y, x'\rangle_{\mathbb{R}^n}\right)\right).$$

Let

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_z, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_z, \quad Z = \partial_z,$$

We observe

$$[X_j, Y_k] = \delta_{jk} Z.$$

The complex gradient

$$W_j = X_j + iY_j.$$

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$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_z, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_z, \quad Z = \partial_z,$$

We observe

$$[X_j, Y_k] = \delta_{jk} Z.$$

The complex gradient

$$W_j = X_j + iY_j.$$

Example: Heisenberg groups

$$\mathbb{H}^n = \{(x, y, z) : x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

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$$L = \sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

By spectral decomposition of the sublaplacian,

$$[W_j, \sqrt{-L}]f = 2i\,\mathcal{T}_j Z f, \quad \forall f \in \mathcal{S}(\mathbb{H}^n)$$

where

$$\mathcal{T}_j = \int_0^{+\infty} y P_y (W_j \sqrt{-L} + \sqrt{-L} W_j) P_y dy.$$

Proposition

Let $1 \leq j \leq n$ *and* $f \in \mathcal{S}(\mathbb{H}^n)$. *Then we have*

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Riesz transform on vector bundles

$\mathbb{M}$: d-dimensional smooth complete Riemannian manifold.

$\mathcal{E}$: finite-dimensional vector bundle over $\mathbb{M}$

$\Gamma(\mathbb{M}, \mathcal{E})$: the space of smooth sections of this bundle.

∇: metric connection on $\mathcal{E}$.

We consider an operator on $\Gamma(\mathbb{M}, \mathcal{E})$

$$\mathcal{L} = \mathcal{F} + \nabla_0 + \sum_{i=1}^{d} \nabla_{X_i}^2,$$

where $\mathcal{F}$ is a smooth symmetric and non positive potential (that is a smooth section of the bundle $\mathbf{End}(\mathcal{E})$).

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where $a_1, \cdots, a_d$ are smooth sections of the bundle $\mathbf{End}(\mathcal{E})$.

Assume

$$d_a \mathcal{L} \eta = \mathcal{L} d_a \eta, \quad \eta \in \Gamma(\mathbb{M}, \mathcal{E}),$$

and

$$\|d_a \eta\|^2 \leq C \sum_{i=1}^{d} \|\nabla_{X_i} \eta\|^2, \quad \eta \in \Gamma(\mathbb{M}, \mathcal{E}).$$

Theorem (Bañuelos-Baudoin-C. 2018)

Let $\eta \in \Gamma_0^\infty(\mathbb{M}, \mathcal{E})$. *Then for* $1 < p < \infty$,

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Gundy-Varopoulos type representation of Riesz transform

Consider the stochastic parallel transport along Y_t, $\theta_t : \mathcal{E}_{Y_t} \to \mathcal{E}_{Y_0}$ and the multiplicative functional $(\mathcal{M}_t)_{t\geq 0}$, solution of the equation

$$\frac{d\mathcal{M}_t}{dt} = \mathcal{M}_t \theta_t \mathcal{F} \theta_t^{-1}, \quad \mathcal{M}_0 = \mathbf{Id}.$$

Let $\eta \in \Gamma_0^\infty(\mathbb{M}, \mathcal{E})$. For almost all $x \in \mathbb{M}$, we have

$$\begin{aligned} & d_a(-\mathcal{L})^{-1/2}\eta(x) \\ & = -2 \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(\theta_\tau^{-1} \mathcal{M}_\tau^* \int_0^\tau (\mathcal{M}_s^*)^{-1} \theta_s d_a Q f(Y_s, B_s) dB_s \mid Y_\tau = x \right). \end{aligned}$$

- **Applications**: Riesz transforms on forms, spinors.

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$\mathbb{M}$: d-dimensional smooth, oriented, complete and stochastically complete Riemannian manifold.

Fermionic construction on the tangent spaces of $\mathbb{M}$.

e_i: local orthonormal frame; θ_i: its dual frame.

The exterior derivative

$$d = \sum_i a_i^* \nabla_{e_i}.$$

The curvature endomorphism (Weitzenböck curvature) is then defined by

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The celebrated Weitzenböck formula writes

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