Maximal parabolic regularity for divergence-form operators with Neumann boundary conditions in rough domains

A. Carbonaro (U. of Genova)

Based on a collaboration with O. Dragičević (U. of Ljubljana)

Harmonic Analysis of Elliptic and Parabolic
Partial Differential Equations

CIRM Luminy April 23, 2018

Elliptic operators in divergence form

$\Omega \subseteq \mathbb{R}^n$ open, $A \in L^\infty(\Omega \to \mathbb{C}^{n,n})$ a strictly elliptic matrix function, i.e. $\exists\ \lambda, \Lambda > 0$ s.t.

$$|\langle A(x)\xi, \eta\rangle| \leqslant \Lambda\, |\xi|\, |\eta|\ , \quad \forall \xi, \eta \in \mathbb{C}^n$$
$$\mathrm{Re}\, \langle A(x)\xi, \xi\rangle \geqslant \lambda |\xi|^2\, , \quad \forall \xi \in \mathbb{C}^n$$

Denote the class of all such matrices by $\mathcal{A}(\Omega)$

Consider the operator

$$L_2^A := -\mathrm{div}\, A\nabla$$

with Neumann boundary conditions on Ω:

L_2^A is the maximal accretive operator on $L^2(\Omega)$ associated with

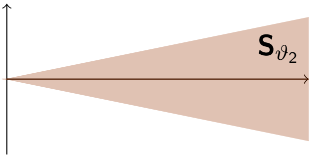
$$\mathfrak{a}(u,v) = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^n}, \quad \mathrm{D}(\mathfrak{a}) = W^{1,2}(\Omega)$$

- L_2^A is **sectorial** with

$$\omega(L_2^A) \leqslant \vartheta_2^* < \pi/2,$$

where ϑ_2^* is the numerical range angle of $\mathfrak{a}$

- $(T_t^A)_{t>0} := (e^{-tL_2^A})_{t>0}$ analytic and contractive in $\mathbf{S}_{\vartheta_2}$



$\vartheta_2^* = \pi/2 - \vartheta_2$

We are interested in:

- Contractivity, analiticity of $(T_t^A)_{t>0}$ in $L^p(\Omega)$ $(p > 1)$

We denote by L_p^A its negative generator

- Bounded H^∞-functional calculus for L_p^A
- Maximal parabolic regularity for L_p^A

We are interested in results that **only depend** on the algebraic properties of the matrix A

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$.

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \operatorname{Re} \langle A(x)\xi, \mathcal{J}_p\xi\rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \mathrm{Re}\, \langle A(x)\xi, \mathcal{J}_p\xi \rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

- $\Delta_2(A) > 0 \iff A$ (uniform strict) elliptic

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \operatorname{Re} \langle A(x)\xi, \mathcal{J}_p \xi \rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

- $\Delta_2(A) > 0 \iff A$ (uniform strict) elliptic
- $|1 - 2/p| < \lambda/\Lambda \Rightarrow \Delta_p(A) > 0$

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \operatorname{Re} \langle A(x)\xi, \mathcal{J}_p\xi \rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

- $\Delta_2(A) > 0 \iff A$ (uniform strict) elliptic
- $|1 - 2/p| < \lambda/\Lambda \Rightarrow \Delta_p(A) > 0$
- $\Delta_p(A) = \Delta_q(A)$

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \mathrm{Re}\,\langle A(x)\xi, \mathcal{J}_p\xi\rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

- $\Delta_2(A) > 0 \iff A$ (uniform strict) elliptic
- $|1 - 2/p| < \lambda/\Lambda \Rightarrow \Delta_p(A) > 0$
- $\Delta_p(A) = \Delta_q(A)$
- $\Delta_p(A) > 0 \iff \Delta_p(A^*) > 0$

p-ellipticity (C.-Dragičević 2015)

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\xi_1 + i\xi_2) = \frac{\xi_1}{p} + i\,\frac{\xi_2}{q}$$

Here $\xi_1, \xi_2 \in \mathbb{R}^n$ and $1/p + 1/q = 1$. For $A \in \mathcal{A}(\Omega)$ set

$$\Delta_p(A) := 2 \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \operatorname{Re} \langle A(x)\xi, \mathcal{J}_p \xi \rangle_{\mathbb{C}^n}.$$

We say that A is p-elliptic if

$$\Delta_p(A) > 0$$

- $\Delta_2(A) > 0 \iff A$ (uniform strict) elliptic
- $|1 - 2/p| < \lambda/\Lambda \Rightarrow \Delta_p(A) > 0$
- $\Delta_p(A) = \Delta_q(A)$
- $\Delta_p(A) > 0 \iff \Delta_p(A^*) > 0$
- $\Delta_p(A) > 0\ \forall p > 1 \iff A$ is real-valued

Main result

Theorem (C.-Dragičević 2018)

Let $\Omega \subseteq \mathbb{R}^n$ be any open set ($n \geqslant 2$). Let $A \in \mathcal{A}(\Omega)$ and $p > 1$. Suppose that $\Delta_p(A) > 0$. Then,

(i) $(T_t^A)_{t>0}$ is analytic and contractive in $L^p(\Omega)$

(ii) L_p^A has maximal parabolic regularity

$\mathcal{I} = \{p \in (1, \infty) : \Delta_p(A) > 0\}$ is an open interval containing 2

$p \in \mathcal{I} \iff q = p' \in \mathcal{I}$

Maximal parabolic regularity

We say that L_p^A has maximal parabolic regularity if for some $r > 1$, all $\tau > 0$ and all $f \in L^r((0,\tau); L^p(\Omega))$, the unique mild solution

$$u(t) := \int_0^t T_{t-s}^A f(s)\,\mathrm{d}s, \quad t \in (0,\tau)$$

to the Cauchy problem

$$\begin{cases} u' + L_p^A u = f; \\ u(0) = 0. \end{cases}$$

belongs to $W^{1,r}(0,\tau); L^p(\Omega)) \cap L^r((0,\tau); \mathrm{D}(L_p^A))$

Maximal parabolic regularity

We say that L_p^A has maximal parabolic regularity if for some $r > 1$, all $\tau > 0$ and all $f \in L^r((0,\tau); L^p(\Omega))$, the unique mild solution

$$u(t) := \int_0^t T_{t-s}^A f(s)\,\mathrm{d}s, \quad t \in (0,\tau)$$

to the Cauchy problem

$$\begin{cases} u' + L_p^A u = f; \\ u(0) = 0. \end{cases}$$

belongs to $W^{1,r}(0,\tau); L^p(\Omega)) \cap L^r((0,\tau); \mathrm{D}(L_p^A))$

Necessary condition: analiticity of $(T_t^A)_{t>0}$ in $L^p(\Omega)$

H^∞ functional calculus

For $\vartheta > \omega(L_p^A)$ and $m \in H^\infty(\mathbf{S}_\vartheta)$ one can define the closed d.d. (possibly unbounded) linear operator $m(L_p^A)$:

$$m(L_p^A)f = \frac{1}{2\pi i}\int_{\partial^+\mathbf{S}_\vartheta} m(z)(z - L_p^A)^{-1} f \, \mathrm{d}z$$

H^∞ functional calculus

For $\vartheta > \omega(L_p^A)$ and $m \in H^\infty(\mathbf{S}_\vartheta)$ one can define the closed d.d. (possibly unbounded) linear operator $m(L_p^A)$:

$$m(L_p^A)f = \frac{1}{2\pi i}\int_{\partial^+\mathbf{S}_\vartheta} m(z)(z - L_p^A)^{-1} f \,\mathrm{d}z$$

We say that L_p^A has bounded $H^\infty(\mathbf{S}_\vartheta)$-calculus if

$$m(L_p^A) \in \mathcal{B}(L^p(\Omega)), \quad \forall m \in H^\infty(\mathbf{S}_\vartheta)$$

H^∞ functional calculus

For $\vartheta > \omega(L_p^A)$ and $m \in H^\infty(\mathbf{S}_\vartheta)$ one can define the closed d.d. (possibly unbounded) linear operator $m(L_p^A)$:

$$m(L_p^A)f = \frac{1}{2\pi i}\int_{\partial^+\mathbf{S}_\vartheta} m(z)(z - L_p^A)^{-1} f \,\mathrm{d}z$$

We say that L_p^A has bounded $H^\infty(\mathbf{S}_\vartheta)$-calculus if

$$m(L_p^A) \in \mathcal{B}(L^p(\Omega)), \quad \forall m \in H^\infty(\mathbf{S}_\vartheta)$$

Functional calculus angle:

$$\omega_H(L_p^A) := \inf\{\vartheta : L_p^A \text{ has bounded } H^\infty(\mathbf{S}_\vartheta)\text{-calculus}\}$$

H^∞ functional calculus

For $\vartheta > \omega(L_p^A)$ and $m \in H^\infty(\mathbf{S}_\vartheta)$ one can define the closed d.d. (possibly unbounded) linear operator $m(L_p^A)$:

$$m(L_p^A)f = \frac{1}{2\pi i}\int_{\partial^+\mathbf{S}_\vartheta} m(z)(z - L_p^A)^{-1} f \,\mathrm{d}z$$

We say that L_p^A has bounded $H^\infty(\mathbf{S}_\vartheta)$-calculus if

$$m(L_p^A) \in \mathcal{B}(L^p(\Omega)), \quad \forall m \in H^\infty(\mathbf{S}_\vartheta)$$

Functional calculus angle:

$$\omega_H(L_p^A) := \inf\{\vartheta : L_p^A \text{ has bounded } H^\infty(\mathbf{S}_\vartheta)\text{-calculus}\}$$

Theorem (Dore-Venni and Prüss-Shor; Kalton-Weis)

$\omega_H(L_p^A) < \pi/2 \;\Rightarrow\; L_p^A$ has maximal parabolic regularity

Reformulation of the main result

Theorem (C.-Dragičević 2018)

Let $\Omega \subseteq \mathbb{R}^n$ be any open set ($n \geqslant 2$). Let $A \in \mathcal{A}(\Omega)$ and $p > 1$. Suppose that $\Delta_p(A) > 0$. Then,

(i) $(T_t^A)_{t>0}$ is analytic and contractive in $L^p(\Omega)$

(ii) $\omega_H(L_p^A) < \pi/2$

- No regularity of $\partial\Omega$, No Sobolev embeddings
- A result of Kunstmann shows that the range in the theorem is optimal for the class $|\Omega| < +\infty$, $A \in \mathcal{A}(\Omega)$ (counterexamples for $\Delta_p(A) \leqslant 0$).

Previous results

REAL COEFFICIENTS

$\Omega \subseteq \mathbb{R}^n$, $A \in \mathcal{A}(\Omega)$, A real-valued

Ouhabaz (1992, 1996): $(T_t^A)_{t>0}$ is sub-Markovian

Kalton-Weis (2001) $\Rightarrow$ $\omega_H(L_p^A) < \pi/2$, for all $p > 1$.

COMPLEX COEFFICIENTS

Define the Sobolev exponents ($n \geqslant 3$)

$$2_\star := \frac{2n}{n+2}, \quad 2^\star := \frac{2n}{n-2}$$

$\mathcal{I}(L^A) := \{p \in [1, \infty] : \sup_{t>0} \| T_t^A \|_p < +\infty\}.$

COMPLEX COEFFICIENTS

Define the Sobolev exponents ($n \geqslant 3$)

$$2_\star := \frac{2n}{n+2}, \quad 2^\star := \frac{2n}{n-2}$$

$\mathcal{I}(L^A) := \{p \in [1, \infty] : \sup_{t>0} \| T_t^A \|_p < +\infty\}.$

(a) $\Omega = \mathbb{R}^n$, $A \in \mathcal{A}(\mathbb{R}^n)$

- Auscher (2004): $(2_\star - \varepsilon, 2^\star + \varepsilon) \subseteq \mathcal{I}(L^A)$, where $\varepsilon = \varepsilon(n, \lambda, \Lambda)$

COMPLEX COEFFICIENTS

Define the Sobolev exponents ($n \geqslant 3$)

$$2_\star := \frac{2n}{n+2}, \quad 2^\star := \frac{2n}{n-2}$$

$$\mathcal{I}(L^A) := \{p \in [1,\infty] : \sup_{t>0} \| T_t^A \|_p < +\infty\}.$$

(a) $\Omega = \mathbb{R}^n$, $A \in \mathcal{A}(\mathbb{R}^n)$

- Auscher (2004): $(2_\star - \varepsilon, 2^\star + \varepsilon) \subseteq \mathcal{I}(L^A)$, where $\varepsilon = \varepsilon(n, \lambda, \Lambda)$
- Hofmann-Mayboroda-McIntosh (2011): this range is sharp for complex A (counterexample for any $p \notin [2_\star, 2^\star]$)

COMPLEX COEFFICIENTS

Define the Sobolev exponents ($n \geqslant 3$)

$$2_\star := \frac{2n}{n+2}, \quad 2^\star := \frac{2n}{n-2}$$

$\mathcal{I}(L^A) := \{p \in [1, \infty] : \sup_{t>0} \| T_t^A \|_p < +\infty\}.$

(a) $\Omega = \mathbb{R}^n$, $A \in \mathcal{A}(\mathbb{R}^n)$

- Auscher (2004): $(2_\star - \varepsilon, 2^\star + \varepsilon) \subseteq \mathcal{I}(L^A)$, where $\varepsilon = \varepsilon(n, \lambda, \Lambda)$
- Hofmann-Mayboroda-McIntosh (2011): this range is sharp for complex A (counterexample for any $p \notin [2_\star, 2^\star]$)

(b) $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz, $A \in \mathcal{A}(\Omega)$

- Egert (2017): $(2_\star - \varepsilon, 2^\star + \varepsilon) \subseteq \mathcal{I}(L^A)$, where $\varepsilon = \varepsilon(n, \lambda, \Lambda, \Omega)$

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L_p^A) = \omega(L_2^A) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ$

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L^A_p) = \omega(L^A_2) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ \supset (2_\star - \varepsilon, 2^\star + \varepsilon)$.

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L_p^A) = \omega(L_2^A) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ \supset (2_\star - \varepsilon, 2^\star + \varepsilon)$.

Also: mixed boundary conditions, systems, Riesz transforms...

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L_p^A) = \omega(L_2^A) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ \supset (2_\star - \varepsilon, 2^\star + \varepsilon)$.

- Earlier results by **Blunck and Kunstmann**
- Fundamental tool: modification of Blunck-Kunstmann weak type (p, p) criterion

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L_p^A) = \omega(L_2^A) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ \supset (2_\star - \varepsilon, 2^\star + \varepsilon)$.

Our result

Let $\Omega \subseteq \mathbb{R}^n$ be any open set. Then $\omega_H(L_p^A) < \pi/2$, for all $p \in (1, \infty)$ s.t. $\Delta_p(A) > 0$.

Theorem (Auscher; Egert)

$\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ bounded and weakly Lipschitz. Then,

$$\omega_H(L_p^A) = \omega(L_2^A) < \pi/2$$

for all $p \in \mathcal{I}(L^A)^\circ \supset (2_\star - \varepsilon, 2^\star + \varepsilon)$.

The proof requires Sobolev embeddings: true because one has Sobolev extension operators, in this case

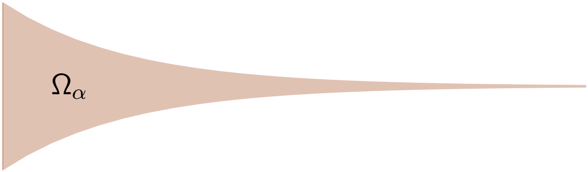
Our result

Let $\Omega \subseteq \mathbb{R}^n$ be any open set. Then $\omega_H(L_p^A) < \pi/2$, for all $p \in (1, \infty)$ s.t. $\Delta_p(A) > 0$.

No Sobolev embeddings and $\omega_H(L_p^A)$ depends on p, in general

An example by Kunstmann

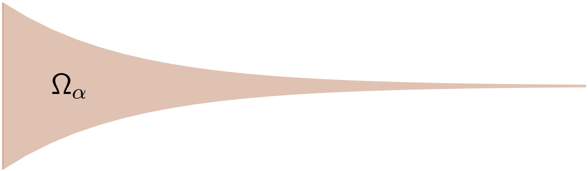
$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : x > 0, |y| < e^{-\alpha x}\}$$



$\Delta^{\Omega_\alpha} := L^I$ the Neumann Laplacian on Ω_α

An example by Kunstmann

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : \; x > 0, \; |y| < e^{-\alpha x}\}$$

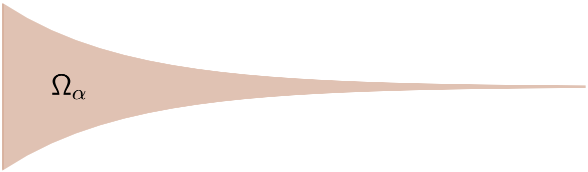


$\Delta^{\Omega_\alpha} := L^I$ the Neumann Laplacian on Ω_α

- Evans and Harris (1989): The resolvent is not compact: No Sobolev embedding of $W^{1,2}(\Omega_\alpha)$ into $L^q(\Omega_\alpha)$, $q > 2$

An example by Kunstmann

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ |y| < e^{-\alpha x}\}$$



$\Delta^{\Omega_\alpha} := L^I$ the Neumann Laplacian on Ω_α

- Evans and Harris (1989): The resolvent is not compact: No Sobolev embedding of $W^{1,2}(\Omega_\alpha)$ into $L^q(\Omega_\alpha)$, $q > 2$
- Davies and Simon (1992): Study of $\sigma(\Delta_2^{\Omega_\alpha})$ by means of a reduction to the Dirichlet operator

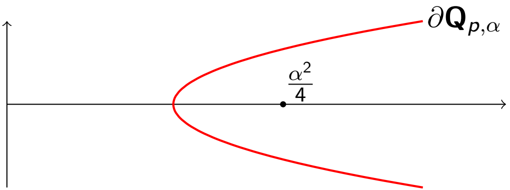
$$-D^2 + \alpha D, \quad \text{on} \quad L^2([0,+\infty); e^{-\alpha x} dx)$$

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$

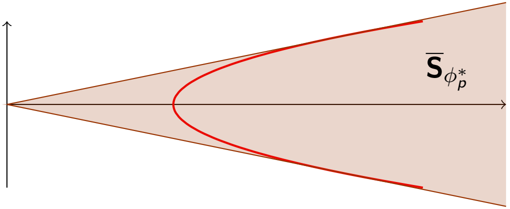


- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



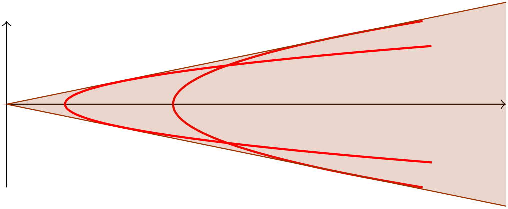
$$\boxed{\phi_p^* = \arcsin |1 - 2/p|}$$

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



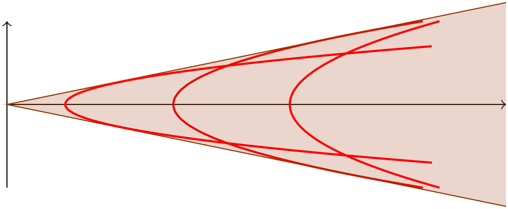
$$\boxed{\phi_p^* = \arcsin |1 - 2/p|}$$

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



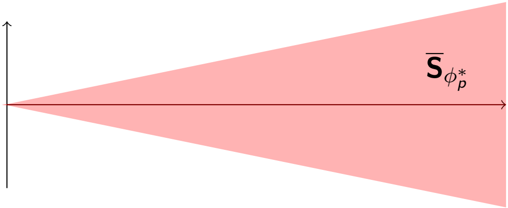
$$\boxed{\phi_p^* = \arcsin |1 - 2/p|}$$

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



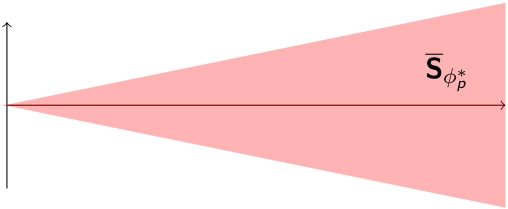
$$\boxed{\phi_p^* = \arcsin |1 - 2/p|}$$

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



$$\boxed{\phi_p^* = \arcsin |1 - 2/p|}$$

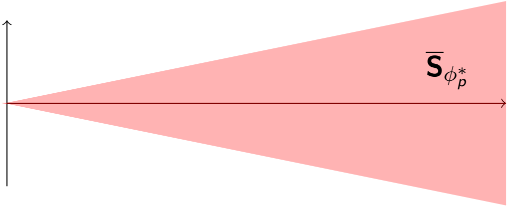
- Kunstmann (2002): $\exists\ \Omega_{\max} \subset \mathbb{R}^2$ of finite measure s.t.
  $\boxed{\sigma(\Delta_p^{\Omega_{\max}}) = \overline{\mathbf{S}}_{\phi_p^*}}$, $1 < p < \infty$.

- Kunstmann (2002): Study of $\sigma_{ess}(\Delta_p^{\Omega_\alpha})$, $p > 1$

$$\{0\} \cup \partial \mathbf{Q}_{p,\alpha} \subseteq \sigma(\Delta_p^{\Omega_\alpha}) \subseteq [0, \alpha^2/4) \cup \partial \mathbf{Q}_{p,\alpha}$$

where $\mathbf{Q}_{p,\alpha} = \alpha^2 \mathbf{Q}_p$ and

$$\mathbf{Q}_p = \left\{ x + iy : x > \frac{p^2}{(p-2)^2} y^2 + \frac{p-1}{p^2} \right\}$$



$$\boxed{\phi_p^* = \arcsin|1 - 2/p|}$$

- Kunstmann (2002): $\exists\ \Omega_{\max} \subset \mathbb{R}^2$ of finite measure s.t. $\boxed{\sigma(\Delta_p^{\Omega_{\max}}) = \overline{\mathbf{S}}_{\phi_p^*}}$, $1 < p < \infty$. This is the maximal L^p-spectrum in the class of generators of symmetric contraction semigroups

$\phi_p^* = \arcsin|1 - 2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p > 1$.

$\phi_p^* = \arcsin|1 - 2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p > 1$.

Theorem (C.-Dragičević 2013)

Let $p > 1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

$\phi_p^* = \arcsin|1-2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p > 1$.

Theorem (C.-Dragičević 2013)

Let $p > 1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

Hence $\boxed{\omega_H(\Delta_p^{\Omega_{\max}}) = \phi_p^*}$

$\phi_p^* = \arcsin|1-2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p > 1$.

Theorem (C.-Dragičević 2013)

Let $p > 1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

Hence $\boxed{\omega_H(\Delta_p^{\Omega_{\max}}) = \phi_p^*}$

$$\omega_H\left(L_p^{e^{i\vartheta}I}\right) = \omega_H\left(e^{i\vartheta}\Delta_p^{\Omega_{\max}}\right) < \pi/2 \iff |\vartheta| < \phi_p$$

$\phi_p^* = \arcsin|1-2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p>1$.

Theorem (C.-Dragičević 2013)

Let $p>1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

Hence $\boxed{\omega_H(\Delta_p^{\Omega_{\max}}) = \phi_p^*}$

$$\omega_H\left(L_p^{e^{i\vartheta}I}\right) = \omega_H\left(e^{i\vartheta}\Delta_p^{\Omega_{\max}}\right) < \pi/2 \iff |\vartheta| < \phi_p$$

$$\boxed{\Delta_p(e^{i\vartheta}I) = \cos\vartheta - \cos\phi_p}$$

$\phi_p^* = \arcsin|1-2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p>1$.

Theorem (C.-Dragičević 2013)

Let $p>1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

Hence $\boxed{\omega_H(\Delta_p^{\Omega_{\max}}) = \phi_p^*}$

$$\omega_H\left(L_p^{e^{i\vartheta}I}\right) = \omega_H\left(e^{i\vartheta}\Delta_p^{\Omega_{\max}}\right) < \pi/2 \iff \Delta_p(e^{i\vartheta}I) > 0$$

$$\boxed{\Delta_p(e^{i\vartheta}I) = \cos\vartheta - \cos\phi_p}$$

$\phi_p^* = \arcsin|1 - 2/p|$, $\phi_p = \pi/2 - \phi_p^*$

Theorem (Bakry '89; Liskevich and Perelmuter '95; Kriegler 2011)

Every symmetric contraction semigroup $(\exp(-t\mathcal{L}))_{t>0}$ is L^p-contractive in $\mathbf{S}_{\phi_p}$. In particular, $\sigma(\mathcal{L}_p) \subseteq \overline{\mathbf{S}}_{\phi_p^*}$ for all $p > 1$.

Theorem (C.-Dragičević 2013)

Let $p > 1$. Then $\omega_H(\mathcal{L}_p) \leqslant \phi_p^*$, for every generator of s.c.s.

Hence $\boxed{\omega_H(\Delta_p^{\Omega_{\max}}) = \phi_p^*}$

$$\omega_H\left(L_p^{e^{i\vartheta}I}\right) = \omega_H\left(e^{i\vartheta}\Delta_p^{\Omega_{\max}}\right) < \pi/2 \iff \Delta_p(e^{i\vartheta}I) > 0$$

$$\boxed{\Delta_p(e^{i\vartheta}I) = \cos\vartheta - \cos\phi_p}$$

Our target: $\Delta_p(A) > 0 \Rightarrow \omega_H(L_p^A) < \pi/2$ (for all Ω and all A)

The example above shows that this range is sharp

Genesis of p-ellipticity

We discovered p-ellipticity by studying "convexity" of power functions

$$F_s(\zeta) = |\zeta|^s, \quad \zeta \in \mathbb{R}^2$$

Motivation: Our interest in "convexity" of the **Nazarov-Treil** (1995) Bellman function $\mathcal{Q}$ which comprises linear combinations of tensor products of power functions

$$\mathcal{Q}(\zeta,\eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q} & ; \ |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \ |\zeta|^p \geqslant |\eta|^q \end{cases}$$

where $p > 2$, $q = p/(p-1)$ and $\delta > 0$.

Genesis of p-ellipticity

We discovered p-ellipticity by studying "convexity" of power functions

$$F_s(\zeta) = |\zeta|^s, \quad \zeta \in \mathbb{R}^2$$

Motivation: Our interest in "convexity" of the **Nazarov-Treil** (1995) Bellman function $\mathcal{Q}$ which comprises linear combinations of tensor products of power functions

$$\mathcal{Q}(\zeta,\eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q} & ; \ |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \ |\zeta|^p \geqslant |\eta|^q \end{cases}$$

where $p > 2$, $q = p/(p-1)$ and $\delta > 0$.

- For small δ the function $\mathcal{Q}$ is uniformly convex

Genesis of p-ellipticity

We discovered p-ellipticity by studying "convexity" of power functions

$$F_s(\zeta) = |\zeta|^s, \quad \zeta \in \mathbb{R}^2$$

Motivation: Our interest in "convexity" of the **Nazarov-Treil** (1995) Bellman function $\mathcal{Q}$ which comprises linear combinations of tensor products of power functions

$$\mathcal{Q}(\zeta,\eta) := |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2|\eta|^{2-q} & ; \ |\zeta|^p \leqslant |\eta|^q \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q & ; \ |\zeta|^p \geqslant |\eta|^q \end{cases}$$

where $p > 2$, $q = p/(p-1)$ and $\delta > 0$.

- For small δ the function $\mathcal{Q}$ is uniformly convex
- We are interested in a different type of convexity related to p-ellipticity: Generalized convexity...later

Dindoš and Pipher (2016)

Independently of us, Dindoš and Pipher discovered the *p-ellipticity* condition and realized it could be used for a new regularity theory for weak solutions to complex coefficient operators

A "replacement" for the De Giorgi-Nash-Moser regularity theory for real coefficients

Dindoš and Pipher (2016)

Independently of us, Dindoš and Pipher discovered the *p-ellipticity* condition and realized it could be used for a new regularity theory for weak solutions to complex coefficient operators

A "replacement" for the De Giorgi-Nash-Moser regularity theory for real coefficients

Application: solvability of the L^p Dirichlet boundary value problem for $\mathrm{div}\,(A\nabla)$

Dindoš and Pipher (2016)

Theorem 1 (Reverse Hölder inequality)

If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to $\mathrm{div}(A\nabla u) = 0$ in Ω and

$$p_0 := \inf\{p > 1 : A \text{ is } p\text{−elliptic}\}$$

then, for any $B_{4r}(x) \subset \Omega$ and $p, q \in (p_0, p_0' n/(n-2))$,

$$\langle |u|^p \rangle^{1/p}_{B_r(x)} \lesssim \langle |u|^q \rangle^{1/q}_{B_{2r}(x)} + \ (E.T.)$$

Constants depend only on n, p_0, Λ.

For elliptic A: counterexample by Mayboroda

for $q = 2$ and any $p > 2n/(n-2)$

Dindoš and Pipher (2016)

Theorem 2 (Caccioppoli estimate)

If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to $\mathrm{div}(A\nabla u) = 0$ in Ω and

$$p_0 := \inf\{p > 1 : A \text{ is } p\text{−elliptic}\}$$

then, for any $B_{4r}(x) \subset \Omega$ and $p \in (p_0, p_0')$,

$$\int_{B_r(x)} |\nabla u|^2 |u|^{p-2} \lesssim r^{-2} \int_{B_{2r}(x)} |u|^p \; + (E.T.)$$

Constants depend only on n, p_0, Λ.

Dindoš and Pipher (2016)

Theorem 2 (Caccioppoli estimate)

If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to $\mathrm{div}(A\nabla u) = 0$ in Ω and

$$p_0 := \inf\{p > 1 : A \text{ is } p\text{−elliptic}\}$$

then, for any $B_{4r}(x) \subset \Omega$ and $p \in (p_0, p_0')$,

$$\int_{B_r(x)} |\nabla u|^2 |u|^{p-2} \lesssim r^{-2} \int_{B_{2r}(x)} |u|^p \; + (E.T.)$$

Constants depend only on n, p_0, Λ.

Consequence:

$$\left[\Delta_p(A) > 0,\ u \in W^{1,2}_{loc}(\Omega) \text{ s.t. } \mathrm{div}\,(A\nabla u) = 0\right] \Rightarrow |u|^{\frac{p-2}{2}} u \in W^{1,2}_{loc}(\Omega)$$

Cialdea-Maz'ya (2005)

p-ellipticity is related to a condition earlier introduced by Cialdea and Maz'ya for studying contractivity of Dirichlet semigroups

Several papers and a book on this subject

Theorem (Cialdea-Maz'ya)

$\Omega \subset \mathbb{R}^n$ bounded of class, say, C^2, $A \in C^1(\overline{\Omega})$. Let $p > 1$. Suppose that for all $x \in \Omega$ and $\alpha, \beta \in \mathbb{R}^n$

$$\frac{4}{pq}\langle \mathrm{Re}\, A(x)\alpha, \alpha\rangle + \langle \mathrm{Re}\, A(x)\beta, \beta\rangle + 2\langle\left(\frac{1}{p}\mathrm{Im}\, A(x) + \frac{1}{q}\mathrm{Im}\, A^*(x)\right)\alpha, \beta\rangle \geqslant 0$$

Then the Dirichlet semigroup $(e^{t\,\mathrm{div}\,(A\nabla)})_{t>0}$ is contractive in L^p

Cialdea-Maz'ya (2005)

p-ellipticity is related to a condition earlier introduced by Cialdea and Maz'ya for studying contractivity of Dirichlet semigroups

Several papers and a book on this subject

Theorem (Cialdea-Maz'ya)

$\Omega \subset \mathbb{R}^n$ bounded of class, say, C^2, $A \in C^1(\overline{\Omega})$. Let $p > 1$. Suppose that for all $x \in \Omega$ and $\alpha, \beta \in \mathbb{R}^n$

$$\frac{4}{pq}\langle \mathrm{Re}\, A(x)\alpha, \alpha\rangle + \langle \mathrm{Re}\, A(x)\beta, \beta\rangle + 2\langle \left(\frac{1}{p}\mathrm{Im}\, A(x) + \frac{1}{q}\mathrm{Im}\, A^*(x)\right)\alpha, \beta\rangle \geqslant 0$$

Then the Dirichlet semigroup $(e^{t\,\mathrm{div}\,(A\nabla)})_{t>0}$ is contractive in L^p

$\Delta_p(A) \geqslant 0 \iff$ Cialdea-Maz'ya condition above

Cialdea-Maz'ya (2005)

p-ellipticity is related to a condition earlier introduced by Cialdea and Maz'ya for studying contractivity of Dirichlet semigroups

Several papers and a book on this subject

Theorem (Cialdea-Maz'ya)

$\Omega \subset \mathbb{R}^n$ bounded of class, say, C^2, $A \in C^1(\overline{\Omega})$. Let $p > 1$. Suppose that for all $x \in \Omega$ and $\alpha, \beta \in \mathbb{R}^n$

$$\frac{4}{pq}\langle \mathrm{Re}\, A(x)\alpha, \alpha\rangle + \langle \mathrm{Re}\, A(x)\beta, \beta\rangle + 2\left\langle\left(\frac{1}{p}\mathrm{Im}\, A(x) + \frac{1}{q}\mathrm{Im}\, A^*(x)\right)\alpha, \beta\right\rangle \geqslant 0$$

Then the Dirichlet semigroup $(e^{t\mathrm{div}\,(A\nabla)})_{t>0}$ is contractive in L^p

C.-Dragičević 2015. $\Delta_p(A) \geqslant 0$ can be interpreted in terms of generalized convexity of the power function F_p

Generalized convexity

Identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathcal{V}(\xi_1 + i\xi_2) = (\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \mathbb{R}^n$$

Given $A \in \mathbb{C}^{n,n}$, consider its real form

$$\mathcal{M}(A) := \mathcal{V}A\mathcal{V}^{-1} = \begin{pmatrix} \mathrm{Re}\, A & -\mathrm{Im}\, A \\ \mathrm{Im}\, A & \mathrm{Re}\, A \end{pmatrix}$$

Fix $\Phi \in C^2(\mathbb{R}^{2k}; \mathbb{R})$. Let $A_1, \dots, A_k \in \mathbb{C}^{n,n}$.

Generalized Hessian of Φ with respect to $A_1, \dots, A_k$:

$$H_\Phi^{(A_1,\dots,A_k)}(\omega) := (\mathcal{M}(A_1) \oplus \cdots \oplus \mathcal{M}(A_k))^T \cdot \Big((d^2\Phi)(\omega) \otimes I_{\mathbb{R}^n}\Big)$$

Definition

We say that Φ is $(A_1, \ldots, A_k)$-convex if

$$H_{\Phi}^{(A_1,\ldots,A_k)}[\omega; X] := \langle H_{\Phi}^{(A_1,\ldots,A_k)}(\omega) X, X \rangle_{\mathbb{R}^{2kn}} \geqslant 0,$$

for all $X \in \mathbb{R}^{2kn}$ and all $\omega \in \mathbb{R}^{2k}$.

Definition

We say that Φ is $(A_1, \dots, A_k)$-convex if

$$H_\Phi^{(A_1,\dots,A_k)}[\omega; X] := \langle H_\Phi^{(A_1,\dots,A_k)}(\omega)X, X\rangle_{\mathbb{R}^{2kn}} \geqslant 0,$$

for all $X \in \mathbb{R}^{2kn}$ and all $\omega \in \mathbb{R}^{2k}$.

Example: Generalized convexity of power functions $F_p(\zeta) = |\zeta|^p$

Let $A \in \mathbb{C}^{n,n}$ and $p > 1$. Then

$$H_{F_p}^A[\zeta; \mathcal{V}(\xi)] = p^2|\zeta|^{p-2}\mathrm{Re}\left\langle A\left(e^{-i\arg(\zeta)}\xi\right), \mathcal{J}_q\left(e^{-i\arg(\zeta)}\xi\right)\right\rangle_{\mathbb{C}^n}$$

$\Delta_p(A) \geqslant 0 \iff F_p$ is A-convex

$\frac{p^2}{2}|\zeta|^{p-2}\Delta_p(A) =$ min. eigenvalue of $(H_{F_p}^A(\zeta) + H_{F_p}^A(\zeta)^T)/2$

Heat-flow monotonicity

Let $\Omega \subseteq \mathbb{R}^n$, $A_1, \ldots, A_k \in \mathcal{A}(\Omega)$ and $\Phi \in C^2(\mathbb{R}^{2k}; \mathbb{R}_+)$.

For $f_1, \ldots, f_k \in (L^p \cap L^q)(\Omega)$, define

$$\boxed{\mathcal{E}(t) := \int_\Omega \Phi\left(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k\right)}$$

Here we identify $T_t^{A_j} f_j$ with $(\mathrm{Re}\, T_t^{A_j} f_j, \mathrm{Im}\, T_t^{A_j} f_j)$

Heat-flow monotonicity

Let $\Omega \subseteq \mathbb{R}^n$, $A_1, \ldots, A_k \in \mathcal{A}(\Omega)$ and $\Phi \in C^2(\mathbb{R}^{2k}; \mathbb{R}_+)$.

For $f_1, \ldots, f_k \in (L^p \cap L^q)(\Omega)$, define

$$\boxed{\mathcal{E}(t) := \int_\Omega \Phi\left(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k\right)}$$

Here we identify $T_t^{A_j} f_j$ with $(\mathrm{Re}\, T_t^{A_j} f_j, \mathrm{Im}\, T_t^{A_j} f_j)$

In applications $\Phi = F_p$ or $\mathcal{Q}$: We need to know a priori that $(T_t^{A_j})_{t>0}$ are analytic in L^p and L^q

Heat-flow monotonicity

Let $\Omega \subseteq \mathbb{R}^n$, $A_1, \ldots, A_k \in \mathcal{A}(\Omega)$ and $\Phi \in C^2(\mathbb{R}^{2k}; \mathbb{R}_+)$.

For $f_1, \ldots, f_k \in (L^p \cap L^q)(\Omega)$, define

$$\boxed{\mathcal{E}(t) := \int_\Omega \Phi\left(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k\right)}$$

Here we identify $T_t^{A_j} f_j$ with $(\mathrm{Re}\, T_t^{A_j} f_j, \mathrm{Im}\, T_t^{A_j} f_j)$

In applications $\Phi = F_p$ or $\mathcal{Q}$: We need to know a priori that $(T_t^{A_j})_{t>0}$ are analytic in L^p and L^q

Set $\zeta^j = \zeta_1^j + i\zeta_2^j$, $j = 1, \ldots, k$ and

$$\partial_{\zeta^j} = \frac{1}{2}\left(\partial_{\zeta_1^j} - i\partial_{\zeta_2^j}\right)$$

a) Suppose that we can interchange derivative and integral:

$$-\mathcal{E}'(t) = \int_\Omega 2\mathrm{Re}\left[\sum_{j=1}^{k}(\partial_{\zeta^j}\Phi)\left(T_t^{A_1}f_1,\ldots,T_t^{A_k}f_k\right)L_2^{A_j}T_t^{A_j}f_j\right]$$

a) Suppose that we can interchange derivative and integral:

$$-\mathcal{E}'(t) = \int_{\Omega} 2\mathrm{Re}\left[\sum_{j=1}^{k} (\partial_{\zeta^j}\Phi)\left(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k\right) L_2^{A_j} T_t^{A_j} f_j\right]$$

b) **Suppose** that

$$\boxed{(\partial_{\zeta^j}\Phi)(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k) \in W^{1,2}(\Omega)}$$

Then the right hand side equals

$$\int_{\Omega} H_{\Phi}^{(A_1,\ldots,A_k)}\left[\left(T_t^{A_1} f_1, \ldots, T_t^{A_k} f_k\right); \left(\nabla T_t^{A_1} f_1, \ldots, \nabla T_t^{A_k} f_k\right)\right]$$

a) Suppose that we can interchange derivative and integral:

$$-\mathcal{E}'(t) = \int_\Omega 2\mathrm{Re}\left[\sum_{j=1}^{k}(\partial_{\zeta^j}\Phi)\left(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k\right)L_2^{A_j}T_t^{A_j}f_j\right]$$

b) **Suppose** that

$$\boxed{(\partial_{\zeta^j}\Phi)(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k)\in W^{1,2}(\Omega)}$$

Then the right hand side equals

$$\int_\Omega H_\Phi^{(A_1,\dots,A_k)}\left[\left(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k\right);\left(\nabla T_t^{A_1}f_1,\dots,\nabla T_t^{A_k}f_k\right)\right]$$

In this case

$$\boxed{\Phi \text{ is } (A_1,\dots,A_k)\text{−convex} \implies \mathcal{E}\searrow \text{ in } (0,+\infty)}$$

a) Suppose that we can interchange derivative and integral:

$$-\mathcal{E}'(t) = \int_\Omega 2\mathrm{Re}\left[\sum_{j=1}^{k}(\partial_{\zeta^j}\Phi)\left(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k\right)L_2^{A_j}T_t^{A_j}f_j\right]$$

b) **Suppose** that

$$\boxed{(\partial_{\zeta^j}\Phi)(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k) \in W^{1,2}(\Omega)}$$

Then the right hand side equals

$$\int_\Omega H_\Phi^{(A_1,\dots,A_k)}\left[\left(T_t^{A_1}f_1,\dots,T_t^{A_k}f_k\right);\left(\nabla T_t^{A_1}f_1,\dots,\nabla T_t^{A_k}f_k\right)\right]$$

Example $\Phi = F_p = |\cdot|^p$, $p > 1$

Assume that $(T_t^A)_{t>0}$ is analytic in $L^p(\Omega)$.

$$f \in L^p(\Omega) \ s.t. \ (\partial_\zeta F_p)(T_t^A f) = p|T_t^A f|^{p-2}\overline{T_t^A f} \in W^{1,2}(\Omega)?$$

a) Suppose that we can interchange derivative and integral:

$$-\mathcal{E}'(t) = \int_\Omega 2\mathrm{Re}\left[\sum_{j=1}^{k}(\partial_{\zeta^j}\Phi)\left(T_t^{A_1}f_1,\ldots,T_t^{A_k}f_k\right)L_2^{A_j}T_t^{A_j}f_j\right]$$

b) **Suppose** that

$$\boxed{(\partial_{\zeta^j}\Phi)(T_t^{A_1}f_1,\ldots,T_t^{A_k}f_k)\in W^{1,2}(\Omega)}$$

Then the right hand side equals

$$\int_\Omega H_\Phi^{(A_1,\ldots,A_k)}\left[\left(T_t^{A_1}f_1,\ldots,T_t^{A_k}f_k\right);\left(\nabla T_t^{A_1}f_1,\ldots,\nabla T_t^{A_k}f_k\right)\right]$$

$\mathrm{D}(L_p^A)$ is unknown, in general. This is a technical problem for us

Semigroup contractivity and analiticity

Target: Prove that given any open $\Omega \subset \mathbb{R}^n$ and any $A \in \mathcal{A}(\Omega)$
$\Delta_p(A) > 0 \Rightarrow (T_t^A)_{t>0}$ analytic and contractive in $L^p(\Omega)$

Semigroup contractivity and analiticity

Target: Prove that given any open $\Omega \subset \mathbb{R}^n$ and any $A \in \mathcal{A}(\Omega)$ $\Delta_p(A) > 0 \Rightarrow (T_t^A)_{t>0}$ analytic and contractive in $L^p(\Omega)$

The proof follows from a result by **Nittka**.

Ouhabaz's invariance criterion

$$B_p := \{u \in L^2 \cap L^p : \|u\|_p \leqslant 1\}, \quad P_p : L^2 \to B_p \ \perp \text{ projection}$$

$(T_t^A)_{t>0}$ contr. in $L^p(\Omega)$ iff $P_p\left(W^{1,2}(\Omega)\right) \subseteq W^{1,2}(\Omega)$ and

$$\mathrm{Re} \int_\Omega \langle A\nabla P_p u, \nabla(u - P_p u)\rangle \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$

Theorem (Nittka 2012)

$\|T_t^A\|_p \leqslant 1$ if and only if

$$\int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$ s.t. $|u|^{p-2}u \in W^{1,2}(\Omega)$

Key tool: Nittka's implicit formula for $P_p : L^2 \to B_p$
(no explicit formula, unless $p = 2, \infty$)

.

Theorem (Nittka 2012)

$\|T_t^A\|_p \leqslant 1$ if and only if

$$\int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$ s.t. $|u|^{p-2}u \in W^{1,2}(\Omega)$

Key tool: Nittka's implicit formula for $P_p : L^2 \to B_p$
(no explicit formula, unless $p = 2, \infty$)

p-**ellipticity implies contractivity:** Suppose that $\Delta_p(A) > 0$.

Theorem (Nittka 2012)

$\|T_t^A\|_p \leqslant 1$ if and only if

$$\int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$ s.t. $|u|^{p-2}u \in W^{1,2}(\Omega)$

Key tool: Nittka's implicit formula for $P_p : L^2 \to B_p$
(no explicit formula, unless $p = 2, \infty$)

***p*-ellipticity implies contractivity:** Suppose that $\Delta_p(A) > 0$.

$$p\mathrm{Re}\,\langle A\nabla v, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} = H^A_{F_p}[u; \nabla u]$$

Theorem (Nittka 2012)

$\|T_t^A\|_p \leqslant 1$ if and only if

$$\int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$ s.t. $|u|^{p-2}u \in W^{1,2}(\Omega)$

Key tool: Nittka's implicit formula for $P_p : L^2 \to B_p$
(no explicit formula, unless $p = 2, \infty$)

***p*-ellipticity implies contractivity:** Suppose that $\Delta_p(A) > 0$.

$$\Delta_p(A) \geqslant 0 \Rightarrow p\mathrm{Re}\,\langle A\nabla v, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

Theorem (Nittka 2012)

$\|T_t^A\|_p \leqslant 1$ if and only if

$$\int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

for all $u \in W^{1,2}(\Omega)$ s.t. $|u|^{p-2}u \in W^{1,2}(\Omega)$

Key tool: Nittka's implicit formula for $P_p : L^2 \to B_p$
(no explicit formula, unless $p = 2, \infty$)

***p*-ellipticity implies contractivity:** Suppose that $\Delta_p(A) > 0$.

$$\Delta_p(A) \geqslant 0 \Rightarrow p\mathrm{Re}\,\langle A\nabla v, \nabla(|u|^{p-2}u)\rangle_{\mathbb{C}^n} \geqslant 0$$

Analiticity: By continuity in ϑ, $\Delta_p(A) > 0$ self-improves into $\Delta_p(e^{i\vartheta}A) > 0$ (for small ϑ)

Back to H^∞-calculus

Target: Prove that given any open $\Omega \subset \mathbb{R}^n$ and any $A \in \mathcal{A}(\Omega)$

$$\Delta_p(A) > 0 \implies \omega_H(L_p^A) < \pi/2,$$

where $\omega_H(L_p^A)$ is the functional calculus angle of L_p^A

Back to H^∞-calculus

Target: Prove that given any open $\Omega \subset \mathbb{R}^n$ and any $A \in \mathcal{A}(\Omega)$

$$\Delta_p(A) > 0 \implies \omega_H(L_p^A) < \pi/2,$$

where $\omega_H(L_p^A)$ is the functional calculus angle of L_p^A

Theorem (Cowling, Doust, McIntosh and Yagi 1996)

Suppose *that for some* $\vartheta \in (0, \frac{\pi}{2})$ *we have*

$$\int_0^\infty \int_\Omega \left|\nabla T_t^{e^{\pm i\vartheta}A} f(x)\right| \left|\nabla T_t^{e^{\mp i\vartheta}A^*} g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \lesssim \|f\|_p \|g\|_q$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. *Then* $\omega_H(L_p^A) < \pi/2$

For $A, B \in \mathcal{A}(\Omega)$ and $p > 1$, define

$$\Delta_p(A, B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2018)

Let $A, B \in \mathcal{A}(\Omega)$ and $p > 1$. Suppose that $\Delta_p(A, B) > 0$. Then

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \lesssim \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

For $A, B \in \mathcal{A}(\Omega)$ and $p > 1$, define

$$\Delta_p(A, B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2018)

Let $A, B \in \mathcal{A}(\Omega)$ and $p > 1$. Suppose that $\Delta_p(A, B) > 0$. Then

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \lesssim \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

The functional calculus result follows from the case $B = A^*$

For $A, B \in \mathcal{A}(\Omega)$ and $p > 1$, define

$$\Delta_p(A, B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2018)

Let $A, B \in \mathcal{A}(\Omega)$ and $p > 1$. Suppose that $\Delta_p(A, B) > 0$. Then

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \, \mathrm{d}x \, \mathrm{d}t \lesssim \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

The functional calculus result follows from the case $B = A^*$

Recall:

- $\Delta_p(A) > 0 \iff \Delta_p(A^*) > 0$
- $\Delta_p(A) > 0$ self-improves into $\Delta_p(e^{i\vartheta} A) > 0$ (for small ϑ)

Square functions

Bilinear integrals are dominated by square functions

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \, \mathrm{d}x \, \mathrm{d}t \leqslant \| G_{L^A}(f) \|_p \| G_{L^B}(g) \|_q$$

$$G_{L^A}(f)(x) := \left(\int_0^\infty \left|\nabla T_t^A f(x)\right|^2 \, \mathrm{d}t \right)^{1/2}$$

Square functions

Bilinear integrals are dominated by square functions

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant \|\mathcal{G}_{L^A}(f)\|_p \|\mathcal{G}_{L^B}(g)\|_q$$

$$\mathcal{G}_{L^A}(f)(x) := \left(\iint_{|x-y|<\sqrt{t}} \left|\nabla T_t^A f(y)\right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n/2}} \right)^{1/2}$$

Square functions

Bilinear integrals are dominated by square functions

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \, \mathrm{d}x \, \mathrm{d}t \leqslant \|\mathcal{G}_{L^A}(f)\|_p \|\mathcal{G}_{L^B}(g)\|_q$$

$$\mathcal{G}_{L^A}(f)(x) := \left(\iint_{|x-y|<\sqrt{t}} \left|\nabla T_t^A f(y)\right|^2 \frac{\mathrm{d}y \, \mathrm{d}t}{t^{n/2}} \right)^{1/2}$$

- Auscher (2004) $\Omega = \mathbb{R}^n$: $\|G_{L^A}\|_p < \infty$ for $p \in (q_-(L^A), q_+(L^A))$
- Auscher, Hofmann, Martell (2012) $\Omega = \mathbb{R}^n$: $\|\mathcal{G}_{L^A}\|_p < \infty$ for $p > p_-(L^A)$, where $(p_-(L^A), p_+(L^A)) = \mathcal{I}(L^A)$

Square functions

Bilinear integrals are dominated by square functions

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant \|\mathcal{G}_{L^A}(f)\|_p \|\mathcal{G}_{L^B}(g)\|_q$$

$$\mathcal{G}_{L^A}(f)(x) := \left(\iint_{|x-y|<\sqrt{t}} \left|\nabla T_t^A f(y)\right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n/2}}\right)^{1/2}$$

- Auscher (2004) $\Omega = \mathbb{R}^n$: $\|G_{L^A}\|_p < \infty$ for $p \in (q_-(L^A), q_+(L^A))$
- Auscher, Hofmann, Martell (2012) $\Omega = \mathbb{R}^n$: $\|\mathcal{G}_{L^A}\|_p < \infty$ for $p > p_-(L^A)$, where $(p_-(L^A), p_+(L^A)) = \mathcal{I}(L^A)$

Under our assumptions boundedness of square functions (vertical/conical) on $L^p(\Omega)$ is unknown.

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_{\Omega} \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Let $p > 2$. Suppose that $\Delta_p(A, B) > 0$. We want to prove that

$$\int_\Omega \left|\nabla T_t^A f\right| \left|\nabla T_t^B g\right| \lesssim -\mathcal{E}'(t)$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_{\Omega} \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Let $p > 2$. Suppose that $\Delta_p(A, B) > 0$. We want to prove that

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f\right| \left|\nabla T_t^B g\right| \lesssim \int_0^\infty -\mathcal{E}'(t) \lesssim \|f\|_p^p + \|g\|_q^q$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Let $p > 2$. Suppose that $\Delta_p(A, B) > 0$. We want to prove that

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f\right| \left|\nabla T_t^B g\right| \lesssim \int_0^\infty -\mathcal{E}'(t) \lesssim \|f\|_p \|g\|_q$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Let $p > 2$. Suppose that $\Delta_p(A, B) > 0$. We want to prove that

$$\int_\Omega \left|\nabla T_t^A f\right| \left|\nabla T_t^B g\right| \lesssim -\mathcal{E}'(t)$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

Bellman-function-heat-flow method

We use a variant of the Bellman-function-heat-flow method introduced by **Petermichl and Volberg (2002)**

Our method consists in studying the strict monotonicity of

$$\mathcal{E}(t) := \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g) \qquad (\Phi = \mathcal{Q})$$

where $\mathcal{Q} = \mathcal{Q}_{p,q,\delta}$ is the Nazarov-Treil function, $p > 2$, $q = p'$

Let $p > 2$. Suppose that $\Delta_p(A, B) > 0$. We want to prove that

$$\int_\Omega \left|\nabla T_t^A f\right| \left|\nabla T_t^B g\right| \lesssim -\mathcal{E}'(t)$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

In principle, this follows from (uniform) generalized convexity of $\mathcal{Q}$

Generalized convexity of $\mathcal{Q}$

$$\Delta_p(A,B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2016)

Let $p \geqslant 2$ and $A, B \in \mathcal{A}(\Omega)$. Suppose that $\Delta_p(A,B) > 0$

Then $\exists \delta > 0$ s.t. $\mathcal{Q}$ is (A,B)-convex in $\mathbb{R}^4 \setminus \Upsilon$: for a.e. $x \in \Omega$

$$H_{\mathcal{Q}}^{(A(x),B(x))}[\omega; (\alpha, \beta)] \gtrsim |\alpha||\beta| \,,$$

for every $\omega \in \mathbb{R}^4 \setminus \Upsilon$ and $\alpha, \beta \in \mathbb{R}^{2n}$.

$$\mathcal{Q} = \begin{cases} F_p \otimes \mathbf{1} + \mathbf{1} \otimes F_q + \delta F_2 \otimes F_{2-q}, & \text{if} \quad |\zeta|^p \leqslant |\eta|^q \\ c_1 F_p \otimes \mathbf{1} + c_2 \mathbf{1} \otimes F_q, & \text{if} \quad |\zeta|^p \geqslant |\eta|^q \end{cases}$$

Generalized convexity of $\mathcal{Q}$

$$\Delta_p(A,B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2016)

Let $p \geqslant 2$ and $A, B \in \mathcal{A}(\Omega)$. Suppose that $\Delta_p(A,B) > 0$

Then $\exists \delta > 0$ s.t. $\mathcal{Q}$ is (A,B)-convex in $\mathbb{R}^4 \setminus \Upsilon$: for a.e. $x \in \Omega$

$$H^{(A(x),B(x))}_{\mathcal{Q}\star\varphi_\nu}[\omega;(\alpha,\beta)] \gtrsim |\alpha||\beta|\,,$$

for every $\omega = (\zeta,\eta) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}^{2n}$.

$$\mathcal{Q} = \begin{cases} F_p \otimes \mathbf{1} + \mathbf{1} \otimes F_q + \delta F_2 \otimes F_{2-q}, & \text{if} \quad |\zeta|^p \leqslant |\eta|^q \\ \\ c_1 F_p \otimes \mathbf{1} + c_2 \mathbf{1} \otimes F_q, & \text{if} \quad |\zeta|^p \geqslant |\eta|^q \end{cases}$$

Generalized convexity of $\mathcal{Q}$

$$\Delta_p(A,B) := \min\{\Delta_p(A), \Delta_p(B)\}$$

Theorem (C.-Dragičević 2016)

Let $p \geqslant 2$ and $A, B \in \mathcal{A}(\Omega)$. Suppose that $\Delta_p(A,B) > 0$

Then $\exists \delta > 0$ s.t. $\mathcal{Q}$ is (A,B)-convex in $\mathbb{R}^4 \setminus \Upsilon$: for a.e. $x \in \Omega$

$$H^{(A(x),B(x))}_{\mathcal{Q}\star\varphi_\nu}[\omega;(\alpha,\beta)] \gtrsim |\alpha||\beta|\,,$$

for every $\omega = (\zeta,\eta) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}^{2n}$.

$$H^{(A,B)}_{\mathcal{Q}}[\omega;(\alpha,\beta)] = \begin{cases} H^A_{F_p}[\zeta;\alpha] + H^B_{F_q}[\eta;\beta] + \delta H^{(A,B)}_{F_2 \otimes F_{2-q}}[\omega;(\alpha,\beta)] \\ \\ c_1 H^A_{F_p}[\zeta;\alpha] + c_2 H^B_{F_q}[\eta;\beta] \end{cases}$$

Bilinear embedding - earlier results

Bilinear embedding - earlier results

$$A = B \text{ real},\ \Omega = \mathbb{R}^n \qquad \text{(Dragičević – Volberg 2011)}$$

Standard convexity of $\mathcal{Q}$

Bilinear embedding - earlier results

$A = B$ real, $\Omega = \mathbb{R}^n$ (Dragičević – Volberg 2011)

$A, B = e^{\pm i\vartheta} I$ (C. – Dragičević 2013)

Related to our universal multiplier theorem
for symmetric contractions

Bilinear embedding - earlier results

$A = B$ real, $\Omega = \mathbb{R}^n$ (Dragičević – Volberg 2011)

$A, B = e^{\pm i\vartheta} I$ (C. – Dragičević 2013)

$A, B = e^{\pm i\vartheta} C$, C real (C. – Dragičević 2015)

Related to our sharp functional calculus result
for nonsymmetric OU operators

$$\Delta_p(e^{i\vartheta}C) > 0 \iff |\vartheta| < \operatorname{arccot} \frac{\sqrt{(p-2)^2 + p^2(\tan\vartheta_C)^2}}{2\sqrt{p-1}}$$

$\vartheta_C :=$ Numerical range angle of C

Bilinear embedding - earlier results

$A = B$ real, $\Omega = \mathbb{R}^n$	(Dragičević – Volberg 2011)
$A, B = e^{\pm i\vartheta} I$	(C. – Dragičević 2013)
$A, B = e^{\pm i\vartheta} C$, C real	(C. – Dragičević 2015)
A, B complex, $\Omega = \mathbb{R}^n$	(C. – Dragičević 2016)

Bilinear embedding - earlier results

$A = B$ real, $\Omega = \mathbb{R}^n$	(Dragičević – Volberg 2011)
$A, B = e^{\pm i\vartheta} I$	(C. – Dragičević 2013)
$A, B = e^{\pm i\vartheta} C$, C real	(C. – Dragičević 2015)
A, B complex, $\Omega = \mathbb{R}^n$	(C. – Dragičević 2016)

Our 2016 proof **does not** extend to arbitrary open $\Omega \subset \mathbb{R}^n$
In this case a modification of our method is needed

Major difficulties in the case $\Omega \neq \mathbb{R}^n$

- $\mathrm{D}(L_p^A)$ is unknown
- No estimates for the kernel of T_t^A
- $T_t^A : \cancel{L^\infty(\Omega) \to L^\infty(\Omega)}$

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

$$\int_0^\infty \int_\Omega \left|\nabla T_t^A f(x)\right| \left|\nabla T_t^B g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \lesssim \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

Remark

Inequality (1) implies dissipativity:

$$(\partial_\zeta \mathcal{Q})(u,0) = C|u|^{p-2}\bar{u}; \quad (\partial_\eta \mathcal{Q})(0,v) = C'|v|^{q-2}\bar{v}$$

“Proof” of the bilinear embedding: $\Delta_p(A, B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u, v) L_2^A u + (\partial_\eta \mathcal{Q})(u, v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

“Proof” of the bilinear embedding: $\Delta_p(A, B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u, v) L_2^A u + (\partial_\eta \mathcal{Q})(u, v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A, B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

$$|(\partial_\zeta \mathcal{Q})(\zeta, \eta)| \lesssim \max\{|\zeta|^{p-1}, |\eta|\}, \quad |(\partial_\eta \mathcal{Q})(\zeta, \eta)| \lesssim |\eta|^{q-1}$$

$$\|d^2 \mathcal{Q}(\zeta, \eta)\| \lesssim |\zeta|^{p-2} + |\eta|^{q-2} + |\eta|^{2-q}$$

"Proof" of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \int_\Omega H_{\mathcal{Q}}^{(A,B)}[(u,v);(\nabla u, \nabla v)] \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

$$|(\partial_\zeta \mathcal{Q})(\zeta,\eta)| \lesssim \max\{|\zeta|^{p-1}, |\eta|\}, \quad |(\partial_\eta \mathcal{Q})(\zeta,\eta)| \lesssim |\eta|^{q-1}$$

$$\|d^2 \mathcal{Q}(\zeta,\eta)\| \lesssim |\zeta|^{p-2} + |\eta|^{q-2} + |\eta|^{2-q}$$

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

$$|(\partial_\zeta \mathcal{Q})(\zeta,\eta)| \lesssim \max\{|\zeta|^{p-1}, |\eta|\}, \quad |(\partial_\eta \mathcal{Q})(\zeta,\eta)| \lesssim |\eta|^{q-1}$$

$$\|d^2 \mathcal{Q}(\zeta,\eta)\| \lesssim |\zeta|^{p-2} + |\eta|^{q-2} + |\eta|^{2-q}$$

"Proof" of the bilinear embedding: $\Delta_p(A, B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u, v) L_2^A u + (\partial_\eta \mathcal{Q})(u, v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A, B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

The case $\Omega = \mathbb{R}^n$.

"Proof" of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

The case $\Omega = \mathbb{R}^n$. We can assume that A, B are smooth:

"Proof" of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$
we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re}\int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

The case $\Omega = \mathbb{R}^n$. We can assume that A, B are smooth:

- $\mathrm{D}(L_p^A) = W^{2,p}(\mathbb{R}^n)$

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re}\int_\Omega (\partial_\zeta \mathcal{Q})(u,v)L_2^A u + (\partial_\eta \mathcal{Q})(u,v)L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ $(\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu)$

The case $\Omega = \mathbb{R}^n$. We can assume that A, B are smooth:

- $\mathrm{D}(L_p^A) = W^{2,p}(\mathbb{R}^n)$
- $C_c^\infty(\mathbb{R}^n)$ is an operator core for L_p^A (the same for L_q^B)

“Proof” of the bilinear embedding: $\Delta_p(A, B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u, v) L_2^A u + (\partial_\eta \mathcal{Q})(u, v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A, B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

The case $\Omega = \mathbb{R}^n$. We can assume that A, B are smooth:

- $\mathrm{D}(L_p^A) = W^{2,p}(\mathbb{R}^n)$
- $C_c^\infty(\mathbb{R}^n)$ is an operator core for L_p^A (the same for L_q^B)
- We reduce the proof of (1) to the case when $u, v \in C_c^\infty(\mathbb{R}^n)$

"Proof" of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re}\int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

The case $\Omega = \mathbb{R}^n$. We can assume that A, B are smooth:

- $\mathrm{D}(L_p^A) = W^{2,p}(\mathbb{R}^n)$
- $C_c^\infty(\mathbb{R}^n)$ is an operator core for L_p^A (the same for L_q^B)
- We reduce the proof of (1) to the case when $u, v \in C_c^\infty(\mathbb{R}^n)$
- In this case the integration by parts in (1) is trivial

“Proof” of the bilinear embedding: $\Delta_p(A,B) > 0$, $p > 2$

By the heat-flow method applied to $\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^A f, T_t^B g)$ we reduce to prove

$$\int_\Omega |\nabla u|\,|\nabla v| \lesssim \mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v \qquad (1)$$

for all $u \in \mathrm{D}(L_p^A) \cap \mathrm{D}(L_q^A)$ and $v \in \mathrm{D}(L_p^B) \cap \mathrm{D}(L_q^B)$

In principle, (1) follows from (A,B)-convexity of $\mathcal{Q}$ ($\mathcal{Q} \leftrightarrow \mathcal{Q} \star \varphi_\nu$)

$$|(\partial_\zeta \mathcal{Q})(\zeta,\eta)| \lesssim \max\{|\zeta|^{p-1}, |\eta|\}, \quad |(\partial_\eta \mathcal{Q})(\zeta,\eta)| \lesssim |\eta|^{q-1}$$

$$\|d^2 \mathcal{Q}(\zeta,\eta)\| \lesssim |\zeta|^{p-2} + |\eta|^{q-2} + |\eta|^{2-q}$$

The general case $\Omega \subset \mathbb{R}^n$

$$\mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

The general case $\Omega \subset \mathbb{R}^n$

$$\mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{R}_n)(u,v) L_2^A u + (\partial_\eta \mathcal{R}_n)(u,v) L_2^B v$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

The general case $\Omega \subset \mathbb{R}^n$

$$\mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{Q})(u,v) L_2^A u + (\partial_\eta \mathcal{Q})(u,v) L_2^B v$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists

The general case $\Omega \subset \mathbb{R}^n$

$$\mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{R}_n)(u,v) L_2^A u + (\partial_\eta \mathcal{R}_n)(u,v) L_2^B v$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists

The general case $\Omega \subset \mathbb{R}^n$

$$\mathrm{Re} \int_\Omega (\partial_\zeta \mathcal{R}_n)(u,v) L_2^A u + (\partial_\eta \mathcal{R}_n)(u,v) L_2^B v$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists
- $\partial_\zeta \mathcal{R}_n(u,v), \partial_\eta \mathcal{R}_n(u,v) \in W^{1,2}(\Omega)$

The general case $\Omega \subset \mathbb{R}^n$

$$\int_\Omega H^{(A,B)}_{\mathcal{R}_n}[(u,v);(\nabla u, \nabla v)]$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists
- $\partial_\zeta \mathcal{R}_n(u,v), \partial_\eta \mathcal{R}_n(u,v) \in W^{1,2}(\Omega)$

The general case $\Omega \subset \mathbb{R}^n$

$$\int_\Omega H_{\mathcal{R}_n}^{(A,B)}[(u,v);(\nabla u, \nabla v)]$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists
- $\partial_\zeta \mathcal{R}_n(u,v), \partial_\eta \mathcal{R}_n(u,v) \in W^{1,2}(\Omega)$
- $d^2\mathcal{R}_n \to d^2\mathcal{Q}$ a.e.

The general case $\Omega \subset \mathbb{R}^n$

$$\int_\Omega H^{(A,B)}_{\mathcal{R}_n}[(u,v);(\nabla u,\nabla v)]$$

Approximation of $\mathcal{Q}$ by a sequence $(\mathcal{R}_n)$ s.t.

- $d\mathcal{R}_n \to d\mathcal{Q}$ and the limit of the integral exists
- $\partial_\zeta \mathcal{R}_n(u,v), \partial_\eta \mathcal{R}_n(u,v) \in W^{1,2}(\Omega)$
- $d^2\mathcal{R}_n \to d^2\mathcal{Q}$ a.e.
- Each $\mathcal{R}_n$ must be (A,B)-convex
  (positivity of the function inside the integral)

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$

2. $\mathcal{R}_n$ is (A, B)-convex

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$
4. $\|d\mathcal{R}_n(\omega)\| \leqslant C(n)|\omega|$

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$
4. $\|d\mathcal{R}_n(\omega)\| \leqslant C(n)|\omega|$
5. $\|d\mathcal{R}_n(\omega)\| \leqslant C(|\omega|^{p-1} + |\omega|^{q-1})$

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$
4. $\|d\mathcal{R}_n(\omega)\| \leqslant C(n)|\omega|$
5. $\|d\mathcal{R}_n(\omega)\| \leqslant C(|\omega|^{p-1} + |\omega|^{q-1})$
6. $d\mathcal{R}_n \to d\mathcal{Q}$ and $d^2\mathcal{R}_n \to d^2\mathcal{Q}$ a.e. in $\mathbb{R}^4$

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$
4. $\|d\mathcal{R}_n(\omega)\| \leqslant C(n)|\omega|$
5. $\|d\mathcal{R}_n(\omega)\| \leqslant C(|\omega|^{p-1} + |\omega|^{q-1})$
6. $d\mathcal{R}_n \to d\mathcal{Q}$ and $d^2\mathcal{R}_n \to d^2\mathcal{Q}$ a.e. in $\mathbb{R}^4$

$\mathcal{R}_n$ depends on A, B and p. The construction of $(\mathcal{R}_n)$ is based on elementary methods but it requires some effort, because (A, B)-convexity is a rigid property.

The general case $\Omega \subset \mathbb{R}^n$. Replace the Bellman function $\mathcal{Q}$ with a sequence of functions $(\mathcal{R}_n)$ s.t.

1. $\mathcal{R}_n \in C^2(\mathbb{R}^4)$
2. $\mathcal{R}_n$ is (A, B)-convex
3. $d^2\mathcal{R}_n \in L^\infty(\mathbb{R}^4; \mathbb{R}^{4,4})$
4. $\|d\mathcal{R}_n(\omega)\| \leqslant C(n)|\omega|$
5. $\|d\mathcal{R}_n(\omega)\| \leqslant C(|\omega|^{p-1} + |\omega|^{q-1})$
6. $d\mathcal{R}_n \to d\mathcal{Q}$ and $d^2\mathcal{R}_n \to d^2\mathcal{Q}$ a.e. in $\mathbb{R}^4$

$\mathcal{R}_n$ depends on A, B and p. The construction of $(\mathcal{R}_n)$ is based on elementary methods but it requires some effort, because (A, B)-convexity is a rigid property.

First step: study of gen. convexity of multivariable power functions (related to (non) contractivity of (T_t^A, T_t^B) on $L^p(\Omega; \mathbb{C}^2)$)

Thank you for your attention!