Algebraically Closed Valued Fields

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Introduction

Today we focus on **algebraically closed valued fields**. This includes here requiring that the valuation is *nontrivial*, that is, the value group is more than just $\{0\}$, equivalently: the valuation ring is not the whole field.

Examples: $\mathbb{C}((t^{\mathbb{Q}}))$. More generally, $\boldsymbol{k}((t^{\Gamma}))$ whenever $\boldsymbol{k}$ is algebraically closed and $\Gamma \neq \{0\}$ is divisible. Also the completion $\mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}_p$.

The algebraically closed case was considered by A. Robinson in the 1950s, then neglected. Since the 1990s it has gained in importance in connection with elimination of imaginaries (Haskell, Hrushovski, Macpherson), "geometric" motivic integration (Hrushovski, Kazhdan), Berkovich geometry, and so on.

Recall that an algebraically closed valued field has henselian valuation ring, algebraically closed residue field and divisible value group. (Converse holds if the residue field has characteristic 0.)

Quantifier Elimination

For QE it is convenient to construe a valued field as a field with a (binary) divisibility relation div on it which corresponds to the valuation as follows:

$$a \operatorname{div} b \;:\Longleftrightarrow\; b \in a\mathcal{O} \;\Longleftrightarrow\; va \leqslant vb.$$

Why not use the unary relation $\mathcal{O}$ instead of the binary relation div? Because $a \operatorname{div} b$ cannot be expressed quantifier-free in terms of the ring operations and $\mathcal{O}$. (Exercise.)

Thus valued fields are L_{div}-structures where $L_{\text{div}} = \{0, 1, -, +, \cdot, \text{div}\}$. Let *ACVF* be a set of axioms in the language L_{div} whose models are exactly the algebraically closed valued fields.

Theorem

ACVF has QE

(A. Robinson: model completeness with a symbol for $\mathcal{O}$ in the language instead of div. Weaker than QE, though Robinson's proof is easy to enhance to give QE.)

Consequences

Corollary

The completions of ACVF are obtained by specifying (characteristic, residue characteristic).

Definability. Let $K \models ACVF$. Then a definable set $S \subseteq K^n$ is a finite union of finite intersections of sets $\{x : f(x) = 0\}$, $\{x : v(f(x)) \leqslant v(g(x))\}$, and $\{x : v(f(x)) > v(g(x))\}$, with $f(T), g(T) \in K[T]$, $T = (T_1, \ldots, T_n)$.

For $n = 1$ and $f(T) \in K[T]$ we have $f(T) = c(T - a_1) \cdots (T - a_d)$ with $c, a_1, \ldots, a_d \in K$, so $v(f(x)) = v(c) + v(x - a_1) + \cdots + v(x - a_d)$ for $x \in K$.

Corollary

A set $S \subseteq K$ is definable iff S is a finite disjoint union of swiss cheeses.

A **swiss cheese** is a set $B \setminus (B_1 \cup \cdots \cup B_m)$, where B is a ball and $B_1, \ldots, B_m$ are disjoint balls properly contained in B. (Here we also count K and one-element subsets of K as balls.)

Sketch of proof of QE

The proof of QE is based on:

- model-theoretic test for QE;
- standard valuation theory.

First item: an L-theory T has QE iff for all models $\mathfrak{M}$ and $\mathfrak{N}$ of T, any embedding of a proper substructure $\mathfrak{A}$ of $\mathfrak{M}$ into $\mathfrak{N}$ can be extended to an embedding of a strictly larger substructure $\mathfrak{B}$ of $\mathfrak{M}$ into some elementary extension of $\mathfrak{N}$.

To apply this for $T =$ACVF, we need to know something about substructures of algebraically closed valued fields: they are integral domains with a "divisibility" relation. One can easily specify the axioms that a "divisibility" relation on an integral domain R should satisfy in order for it to extend (uniquely) to the divisibility relation on $\text{Frac}(R)$ corresponding to a valuation.

Relevant valuation theory

At this stage we need a result which I state here without proof:

Theorem

Let K be a valued field with valuation ring $\mathcal{O}$ and K^a an algebraic closure of K. Then there is a valuation ring $\mathcal{O}^a$ of K^a such that $\mathcal{O}^a \cap K = \mathcal{O}$; moreover, any two such valuation rings $\mathcal{O}^a$ are conjugate under the action of $\mathrm{Gal}(K^a|K)$.

Remark: $\mathcal{O}$ being *henselian* is equivalent to $\mathcal{O}^a$ as above being unique. This is important in the AKE-story, but we won't need it. *Model-theoretic consequence*: the definably closed subsets of models of ACVF are exactly the perfect subfields whose valuation ring is henselian.

For our purpose we now understand enough about extending valuations to the algebraic closure. Let us turn to extending a valuation on K to an extension $K(x)$ with x transcendental over K. We need to consider only three kinds of such extensions.

Non-immediate transcendental extensions

The first kind increases the residue field:

Lemma

Let $\mathcal{O}$ be a valuation ring of K. Then there is a unique valuation ring $\mathcal{O}(x)$ of $K(x)$ such that $(K, \mathcal{O}) \subseteq (K(x), \mathcal{O}(x))$, $x \in \mathcal{O}(x)$, and the residue class of x is transcendental over $\boldsymbol{k} = \mathcal{O}/\mathfrak{m}$, namely

$$\mathcal{O}(x) \;:=\; \{f(x)/g(x) : \; f(x), g(x) \in \mathcal{O}[x], \; g(x) \notin \mathfrak{m}\mathcal{O}[x]\}$$

The second kind increases the value group:

Lemma

Let $v : K^\times \to \Gamma$ be a valuation and let $\Gamma \subseteq \Gamma + \mathbb{Z}\alpha$ be an ordered abelian group extension with $n\alpha \notin \Gamma$ for all $n \geqslant 1$. Then v extends uniquely to a valuation $v_\alpha : K(x)^\times \to \Gamma + \mathbb{Z}\alpha$ such that $v_\alpha(x) = \alpha$.

Immediate transcendental extensions

Third kind: the extension is immediate. Here the relevant fact is:

Lemma

If $(K, \mathcal{O}) \subseteq (K(x), \mathcal{O}_x)$ is an immediate valued field extension, then for any $a_1, \ldots, a_n \in K$ there exists $a \in K$ such that $v(x - a_1) = v(a - a_1), \ldots, v(x - a_n) = v(a - a_n)$.

Proof of QE

To prove QE for ACVF, one reduces to the case where we have models $\mathfrak{M}$ and $\mathfrak{N}$ of ACVF, a substructure K of $\mathfrak{M}$ with $K \neq \mathfrak{M}$ and an embedding $i : K \to \mathfrak{N}$. We need to extend this embedding to a strictly larger substructure L of $\mathfrak{M}$ into some elementary extension of $\mathfrak{N}$. If the underlying ring of K is not yet a field, we pass to the fraction field of K and $i(K)$ in both $\mathfrak{M}$ and $\mathfrak{N}$. So we can assume K is a valued subfield of $\mathfrak{M}$. Using the result on extending the valuation to K^a we can even assume that K is algebraically closed. (But the valuation of K might still be trivial.)

Case 1: the residue field of $\mathfrak{M}$ is strictly larger than the residue field of K. Then we take $x \in \mathfrak{M}$ with $vx = 0$ such that $\operatorname{res}(x) \notin \operatorname{res}(K)$, so $\operatorname{res}(x)$ is transcendental over $\operatorname{res}(K)$. After passing to an elementary extension of $\mathfrak{N}$, if necessary, we can take $y \in \mathfrak{N}$ with $vy = 0$ such that $\operatorname{res}(y)$ is transcendental over $\operatorname{res} i(K)$. Then the first lemma yields an extension of i to an embedding $K(x) \to \mathfrak{N}$ sending x to y.

Proof of QE, continued

Case 2: the value group of $\mathfrak{M}$ is strictly larger than the value group of K. Proceed as in Case 1, using now the second lemma instead of the first: take $x \in \mathfrak{M}$ such that $\alpha := vx \notin \Gamma := v(K^\times)$. Since Γ is divisible, we have $n\alpha \notin \Gamma$ for all $n \geqslant 1$, and so on ...

Case 3: $\mathfrak{M}$ is an immediate extension of K. Then take any $x \in \mathfrak{M}$, $x \notin K$. After passing to an elementary extension of $\mathfrak{N}$, if necessary, the third lemma gives an element y in $\mathfrak{N}$ such that $v(y-a) = v(x-a)$ for all $a \in K$, where for simplicity of notation we identify K with iK via i. This yields an extension of i to an embedding $K(x) \to \mathfrak{N}$ sending x to y.

This finishes the proof. Byproduct of the proof and the first two lemmas: if K is an algebraically closed valued field, then K has an elementary extension with strictly larger residue field but the same value group, and also an elementary extension with strictly larger value group but the same residue field. We shall use this fact at the end.

Induced structure on the residue field and value group

Let K be an algebraically closed valued field. Simple model-theoretic arguments yield:

Theorem

If $X \subseteq K^n$ is definable, then $\mathrm{res}(X) \subseteq \boldsymbol{k}^n$ is constructible and $v(X) \subseteq \Gamma^n$ is semilinear.

The residue field $\boldsymbol{k}$ and the value group Γ do not interact; they are *orthogonal*:

Theorem

Let $X \subseteq K^{m+n}$ be definable. Then its image in $\boldsymbol{k}^m \times \Gamma^n$ is a finite union of sets $Y \times Z$ with constructible $Y \subseteq \boldsymbol{k}^m$ and semilinear $Z \subseteq \Gamma^n$.

For example, there are no definable maps $\boldsymbol{k}^m \to \Gamma^n$ with infinite image. (Here "definable" means that the graph of the map is the image in $\boldsymbol{k}^m \times \Gamma^n$ of a definable subset of K^{m+n}.) Likewise, there are no definable maps $\Gamma^n \to \boldsymbol{k}^m$ with infinite image.

Can we "eliminate" the valuation?

Is there a definable map $f : K \to K^n$ such that for all x, y,

$$f(x) = f(y) \iff v(x) = v(y)?$$

Background to the question. It is well-known that if $F \models ACF$ and E is a definable equivalence relation on a definable set $X \subseteq F^m$, then there is a definable map $f : X \to F^n$ such that for all $x, y \in X$, $f(x) = f(y) \iff xEy$, so that f induces a bijection $X/E \to f(X)$. It means that the abstract "definable" quotient set X/E can be represented by the more geometric (constructible) set $f(X) \subseteq K^n$.

In the situation above, we have the definable equivalence relation "$v(x) = v(y)$" on K.

Can we "eliminate" the valuation?

Fact: there is no such definable map f.

Proof: Suppose towards a contradiction that $f : K \to K^n$ is definable and for all $x, y \in K$, $f(x) = f(y) \iff v(x) = v(y)$. This remains true in passing to an elementary extension of K. By an earlier remark we can do this in such a way that the value group does not change but the field K becomes strictly bigger, and eventually of larger cardinality than Γ. Thus

$$\mathrm{card}(K) \;>\; \mathrm{card}(\Gamma) \;=\; \mathrm{card}(f(K)).$$

Now $f(K) \subseteq K^n$ is infinite, so one of its coordinate projections $\pi(f(K)) \subseteq K$ is infinite, so $\pi(f(K))$ contains an infinite swiss cheese. But an infinite swiss cheese has nonempty interior in K, and so its cardinality equals that of K, contradicting that $\mathrm{card}(f(K)) < \mathrm{card}(K)$.

The residue map cannot be eliminated either

In the same way it follows that there is no definable map $f : \mathcal{O} \to K^n$ such that for all $x, y \in \mathcal{O}$, $f(x) = f(y) \iff \text{res}(x) = \text{res}(y)$.

Even better: there is no definable map $f : K \to K^n \times \boldsymbol{k}^m$ such that for all $x, y \in K$, $f(x) = f(y) \iff v(x) = v(y)$.

Likewise, there is no definable map $f : \mathcal{O} \to K^n \times \Gamma^m$ such that for all $x, y \in \mathcal{O}$, $f(x) = f(y) \iff \text{res}(x) = \text{res}(y)$.