Valued fields III
Elimination of imaginaries in algebraically closed valued fields

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EI

Definition

A complete theory T has *elimination of imaginaries* if, for all $M \models T$, for all $n > 0$ and for all $\emptyset$-definable equivalence relations E on M^n there are $m > 0$ and an $\emptyset$-definable function f_E such that for all $x, y \in M^n$

$$f_E(x) = f_E(y) \Longleftrightarrow xEy.$$

An *imaginary* is an equivalence class of a $\emptyset$-definable equivalence relation.

Any imaginary a/E is an $\{a\}$-definable set, and hence, if T has quantifier elimination, is well understood. But M^n/E is not necessary definable, hence QE does not help to understand the quotient structure. If T has elimination of imaginaries, then M^n/E is identified with $f_E(M^n)$, which is a definable subset of M^m.

EI

Fix language $\mathcal{L}$ for theory T, and sufficiently saturated model $\mathcal{U}$.
For any $\sigma \in \mathrm{Aut}(\mathcal{U})$, and any imaginary $e = a/E$ of an $\emptyset$-definable equivalence relation E,

$$\sigma(a/E) = a/E \text{ if and only if } \sigma(f_E(a)) = f_E(a).$$

We call $f_E(a)$ a *code* for a/E. Observe that $f_E(a) \in \mathrm{dcl}(e)$ and $e \in \mathrm{dcl}^{\mathrm{eq}}(f_E(a))$.
More generally, for any definable set X, we write $\ulcorner X \urcorner$ for a (tuple of) elements with the property
$\sigma(X) = X$ (setwise) if and only if $\sigma(\ulcorner X \urcorner) = \ulcorner X \urcorner$ (pointwise) for all σ.
In a theory with at least two definable elements, elimination of imaginaries is equivalent to saying every definable set has a finite code.
Working in $\mathcal{L}^{\mathrm{eq}}$, $e = a/E$ is an element of $\mathcal{U}^{\mathrm{eq}}$, so automatically we have EI (e codes itself).
EI: Find a (hopefully controllable) fragment of $\mathcal{L}^{\mathrm{eq}}$ in which every definable set has a finite code.

Examples of EI

- An infinite pure set in the language of equality does not have EI. A finite set with more than one element does not have a code.
- An algebraically closed field in the language of rings does have EI.
  - A finite set $\{a_1, \ldots, a_n\}$ is coded by the tuple of coefficients of the polynomial $\prod_{i=1}^{n}(X - a_i)$.
  - A general definable set is, by quantifier elimination, a boolean combination of Zariski-closed sets. Each Zariski-closed set is coded because it has a unique smallest field of definition (equivalently, by coding the ideal of polynomials that vanish on the Zariski-closed set).

Examples of EI

- An algebraically closed valued field K in the one-sorted language of valued fields does not have EI.
  - $xEy \iff (v(x) = v(y))$ cannot be eliminated (set of equivalence classes is the value group)
  - $xEy \iff y(x) = v(y) = 0 \,\&\, v(x - y) > 0$ cannot be eliminated (set of equivalence classes is the residue field)
  - $(x_1, x_2)E(y_1, y_2) \iff B_{\geq v(x_1 - x_2)}(x_1) = B_{\geq v(y_1 - y_2)}(y_1)$ cannot be eliminated (set of equivalence classes is coded by the set of all closed balls)

More about balls in ACVF: closed ball containing 0

$$\begin{aligned} B_{\geq\gamma}(0) &= \{x \in K : v(x) \geq \gamma\} \\ &= \{x \in K : \exists r \in \mathcal{O}_K (x = rc)\} \text{ for any fixed } c \text{ with } v(c) = \gamma \\ &= \gamma \mathcal{O}_K \end{aligned}$$

Thus $B_{\geq\gamma}(0)$ is interdefinable with $[c]_E$, where cEc' iff $v(c) = v(c')$; that is, c/c' is invertible in $\mathcal{O}_K$.

Also, $B_{\geq\gamma}(0)$ has the algebraic structure of an $\mathcal{O}_K$-module.

More about balls in ACVF: closed ball not containing 0

$$B_{\geq\gamma}(a) = \{x \in K : v(x-a) \geq \gamma\}$$

is not an $\mathcal{O}_K$-module, but is a *torsor*: if $x, y, z \in B_{\geq\gamma}(a)$ then $x - y + z \in B_{\geq\gamma}(a)$. Consider the $\mathcal{O}_K$-module generated by $\{1\} \times B_{\geq\gamma}(a)$:

$$\begin{aligned} L &= \langle \{1\} \times B_{\geq\gamma}(a) \rangle \\ &= \{(x,y) \in K \times K : \exists (r,s) \in \mathcal{O}_K \times \mathcal{O}_K \big((x,y) = (r+s, rc_1 + sc_2)\big)\} \\ &\qquad\qquad \text{for some fixed } c_1, c_2 \text{ in } B_{\geq\gamma}(a) \\ &= \{(x,y) \in K \times K : \exists (r,s) \in \mathcal{O}_K \times \mathcal{O}_K \big((x,y) = (r,s) \begin{pmatrix} 1 & c_1 \\ 1 & c_2 \end{pmatrix}\big)\} \end{aligned}$$

Thus $B_{\geq\gamma}(a)$ is interdefinable with L which is interdefinable with $[(c_1, c_2)]$ under the equivalence relation $(c_1, c_2)E(c_1', c_2')$ if and only if $\begin{pmatrix} 1 & c_1 \\ 1 & c_2 \end{pmatrix} \begin{pmatrix} 1 & c_1' \\ 1 & c_2' \end{pmatrix}^{-1}$ is invertible in $\mathrm{GL}_2(\mathcal{O}_K)$.

More imaginaries in ACVF

More generally, every freely generated rank 2 $\mathcal{O}_K$-module of K^2 is interdefinable with an equivalence class of an $\emptyset$-definable equivalence relation.

More generally, every freely generated rank n $\mathcal{O}_K$-module of K^n is interdefinable with an equivalence class of an $\emptyset$-definable equivalence relation.

More about balls in ACVF: open balls

Open ball 'on the spine':

$$\begin{aligned} B_{>\gamma}(c) &= \{x \in K : v(x-c) > \gamma = v(c)\} \\ &= c + \{x \in K : v(x) > \gamma = v(c)\} \\ &= c + \gamma\mathfrak{m}_K \end{aligned}$$

Thus $B_{>\gamma}(c) \in \gamma\mathcal{O}_K/\gamma\mathfrak{m}_K$. Notice that $\gamma\mathcal{O}_K/\gamma\mathfrak{m}_K$ has the structure of a k_K-vector space.

Open ball 'off the spine':

$$\begin{aligned} B_{>\gamma}(a) &= \{x \in K : v(x-a) > \gamma > v(a)\} \\ &= a + \{x \in K : v(x) > \gamma\} \end{aligned}$$

$B_{>\gamma}(a)$ is interdefinable with an element of $L/\mathfrak{m}L$, where $L = \langle \{1\} \times B_{\geq\gamma}(a) \rangle$.

Notation

Valued field K has

- valuation ring $\mathcal{O}_K = \{x \in K : v(x) \geq 0\}$,
- with units $\mathcal{O}_K^\times = \{x \in \mathcal{O} : v(x) = 0\}$, and
- maximal ideal $\mathfrak{m}_K = \{x \in \mathcal{O} : v(x) > 0\}$.
- The value group is $\Gamma_K = K^\times / \mathcal{O}^\times$ and
- the residue field is $k_K = \mathcal{O}_K / \mathfrak{m}_K$.

Geometric sorts: the lattices

S_1

Pick generator $c \in K$, take $\Lambda_c = \{x \in K : \exists\, r \in \mathcal{O}_K (x = rc)\}$.
$\Lambda_c = \Lambda_{c'} \iff v(c) = v(c') \iff c'c^{-1} \in \mathcal{O}_K^\times$

S_n

Pick generator $C \in K^{n^2}$, take $\Lambda_C = \{x \in K^n : \exists\, r \in \mathcal{O}_K^n (x = rC)\}$.
$\Lambda_C = \Lambda_{C'} \iff C'C^{-1} \in \mathrm{GL}_n(\mathcal{O}_K)$

Geometric sorts: the torsors

S_1

Pick generator $c \in K$, take $\Lambda_c = \{x \in K : \exists\, r \in \mathcal{O}_K(x = rc)\}$.
$\Lambda_c = \Lambda_{c'} \iff v(c) = v(c') \iff c'c^{-1} \in \mathcal{O}_K^{\times}$

T_1

$\mathrm{res}(\Lambda) = \Lambda/\mathfrak{m}\Lambda = \{rc + c\mathfrak{m} : r \in \mathcal{O}_K\}$ is a one-dimensional k_K-vector space
$T_1 = \bigcup_{\Lambda \in S_1} \mathrm{res}(\Lambda)$

S_n

Pick generator $C \in K^{n^2}$, take $\Lambda_C = \{x \in K^n : \exists\, r \in \mathcal{O}_K^n(x = rC)\}$.
$\Lambda_C = \Lambda_{C'} \iff C'C^{-1} \in \mathrm{GL}_n(\mathcal{O}_K)$

T_n

$\mathrm{res}(\Lambda) = \Lambda/\mathfrak{m}\Lambda = \{rC + C\mathfrak{m} : r \in \mathcal{O}_K^n\}$ is an n-dimensional k_K-vector space
$T_n = \bigcup_{\Lambda \in S_n} \mathrm{res}(\Lambda)$

Elimination of imaginaries in ACVF

Geometric sorts

$\mathcal{S} = \bigcup_n S_n$, where S_n is the set of free $\mathcal{O}_K$-submodules of K^n of rank n (a *lattice*)
$\mathcal{T} = \bigcup_n T_n$, where
$T_n = \bigcup_{\Lambda \in S_n} \mathrm{res}\Lambda = \bigcup_{\Lambda \in S_n} \Lambda / \mathfrak{m}\Lambda = \{(\Lambda, \xi) : \Lambda \in S_n, \xi \in \Lambda\}$

Theorem 1

The theory ACVF has elimination of imaginaries in the geometric sorts.

Some history

- (1993) A. Macintyre, P. Scowcroft: Extending the language in natural ways does not suffice to eliminate imaginaries in pCF
- (1995) J. Holly: sorts for the balls suffice to eliminate one variable imaginaries in ACVF and RCVF
- (1997) P. Scowcroft: the three-sorted language (K, Γ, k) does not suffice for EI in pCF
- (2006) D. Haskell, E. Hrushovski, H.D. Macpherson: ACVF has EI in the geometric sorts (and not in any finite subset)
- (2006) T. Mellor: RCVF has EI in the geometric sorts
- (2006/2015) E. Hrushovski, B. Martin, S. Rideau: pCF has EI in the geometric sorts (only need $\mathcal{S}$) (uniformly in p)
- (2014) W. Johnson: a smoother proof that ACVF has EI
- (2015) S. Rideau: $\mathrm{VDF}_{\mathcal{EC}}$ has EI in the geometric sorts
- (2016) M. Hils, M. Kamensky, S. Rideau: SCVF_e has EI in the geometric sorts

Criterion for EI

Theorem 2

Let T be a theory in a language with home sort K and let $\mathcal{G}$ be a collection of sorts from K^{eq}. Suppose that:

1. for every non-empty definable set X in K^1 there is an $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable type in X;
2. every definable type in K^n has a code (possibly infinite) in $\mathcal{G}$;
3. every finite set of tuples from $\mathcal{G}$ has a code in $\mathcal{G}$.

Then T has elimination of imaginaries in the sorts $\mathcal{G}$.

Note: the conditions are sufficient but not necessary; 1) fails in $\mathbb{Q}_p$.

Comments on the proof of the EI criterion

First show by induction on n that for every non-empty definable set X in K^n there is an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable type in X.

Project onto the first $n-1$ coordinates, by inductive hypothesis there is a $\operatorname{acl}^{\mathrm{eq}}(\ulcorner \pi(X) \urcorner)$-definable type in $\pi(X)$. Take a realisation a_1 of this type, look at the fibre above it which is a $\ulcorner X \urcorner a_1$-definable subset of K^1 so by assumption 1) contains an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner X \urcorner a_1)$-definable type. Take a realisation a_2. Argue that there is an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable type realised by (a_1, a_2) and hence contained in X.

Now take e to be any imaginary, X a definable set for which it is the code, p an $\operatorname{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable type in X and t the code from $\mathcal{G}$ given by 2) for p. Then $e \in \operatorname{dcl}^{\mathrm{eq}}(t)$ and $t \in \operatorname{acl}^{\mathcal{G}}(e)$. By compactness, can cut t down to a finite set t_0. Since the set of conjugates of t_0 is finite, by 3) it is coded in $\mathcal{G}$, say by s. Then $s \in \operatorname{dcl}^{\mathcal{G}}(e)$ and $e \in \operatorname{dcl}^{\mathrm{eq}}(s)$, as required.

definable types in ACVF

Recall that a type p is definable if, for any formula $\varphi(x, b)$ there is a formula d_φ such that $\varphi(x, b) \in p$ if and only if $d_\varphi(b)$.

- The generic type of an open or closed ball is definable. Let B be an open or closed ball defined over C. $p_B(x)$ says that $x \in B$ and for any C-definable ball B' with $B' \subset B$, $x \notin B'$. Any formula $\varphi(x, b)$ is in p_B provided any realisation of p_B is in the set defined by φ. As this set is a finite union of Swiss cheeses, this will hold provided B is in one of the balls of the Swiss cheese, and not in one of its holes. This depends only on the parameters b.
- The generic type of the residue field is definable. $p_k(x)$ says that $x \in k$ and x is not algebraic. This is definable because k is strongly minimal.

more definable types in ACVF

- Consider the generic type $p_{\mathcal{O}}$ of the valuation ring. Every proper sub-ball of $\mathcal{O}$ is contained in a unique ball of the form $\mathrm{res}^{-1}(\alpha)$ for $\alpha \in k$. So

  $$x \models p_{\mathcal{O}}|C \iff \mathrm{res}(x) \models p_k|C$$

  We say that $p_{\mathcal{O}}$ is *dominated along* res and write $p_k = \mathrm{res}_* p_{\mathcal{O}}$. p_k is *generically stable*, and hence so also is $p_{\mathcal{O}}$.
- Because $p_{\mathcal{O}}$ is generically stable, $p_{\mathcal{O}}^{\otimes n} = p_{\mathcal{O}^n}$ is well-defined and generically stable. $\mathrm{res}_* p_{\mathcal{O}^n} = p_{k^n}$ which is stabilized by the action of $\mathrm{GL}_n(k)$, so $p_{\mathcal{O}^n}$ is stabilized by $\mathrm{GL}_n(\mathcal{O})$. So for any lattice Λ we can define the generic type of Λ, $p_\Lambda = g_* p_{\mathcal{O}^n}$ where g is any linear transformation sending $\mathcal{O}^n$ to Λ. p_Λ is $\ulcorner \Lambda \urcorner$-definable.

Satisfying the conditions of the EI criterion: 1)

For every non-empty definable set X in K^1 there is an $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$-definable type in X.

This follows easily from quantifier elimination and the above discussion of definable types. X is a disjoint union of Swiss cheeses $B \setminus B_1 \cup \cdots \cup B_n$, which are $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$-definable. For any of the balls B in the union, the generic type p_B is in B, hence in X, and is $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$-definable.

Satisfying the conditions of the EI criterion: 2)

Every definable type in K^n has a code (possibly infinite) in $\mathcal{G}$.

By quantifier elimination, any definable type p in K^n is determined by formulas of the form

$$Q(x) = 0 \qquad \text{and} \qquad v(Q_1(x)) \leq v(Q_2(x))$$

where Q, Q_1, Q_2 are polynomials over K in n variables.
Let V_d be the set of polynomials of degree less than or equal to d (in a fixed number of variables) and observe that V_d is a K-vector space. Let

$$I_d = \{Q \in V_d : Q(x) = 0 \in p\}$$
$$R_d = \{(Q_1, Q_2) \in V_d \times V_d : v(Q_1(x)) \leq v(Q_2(x)) \in p\}.$$

I_d is a subspace of V_d, so I_d is coded by a tuple from K
V_d/I_d has a $\ulcorner I_d \urcorner$-definable basis, so is $\ulcorner I_d \urcorner$-definably isomorphic to K^m for some m.

Satisfying the conditions of the EI criterion: 2)

R_d induces a *valued vector space* structure on V_d/I_d and therefore has a code r_d in $\mathcal{G}$ by:

Theorem 3

The code for a definable valued vector space structure on K^m is interdefinable with an element of the geometric sorts.

The sequence $(\ulcorner I_d \urcorner, r_d)_{d>0}$ is the required code in $\mathcal{G}$ for p.

Satisfying the conditions of the EI criterion: 3)

Fix one of the geometric sorts G. A finite set from G is an element of $\mathrm{Sym}^n(G)$ (the n-fold symmetric product of G). We need to code elements of $\mathrm{Sym}^n(G)$.

Suppose we can find some other G' and a map $\pi : G' \to G$ such that

- for every element $b \in G$, $\pi^{-1}(b)$ contains a b-definable generically stable type;
- G' embeds into $K^m \times K^\ell$ for some m, ℓ.

Lemma 1

Definable types in $\mathrm{Sym}^n(K^m \times K^\ell)$ have geometric codes.

By the lemma, definable types in $\mathrm{Sym}^n(G')$ have geometric codes, hence definable types in $\mathrm{Sym}^n(G)$ have geometric codes (this uses the first condition on π). In particular, elements (constant types) in $\mathrm{Sym}^n(G)$ have geometric codes.

Revision of the geometric sorts

Geometric sorts

$\mathcal{S} = \bigcup_n S_n$, where S_n is the set of free $\mathcal{O}_K$-submodules Λ of K^n of rank n (a *lattice*)

$\mathcal{T} = \bigcup_n T_n$, where

$T_n = \bigcup_{\Lambda \in S_n} \mathrm{res}\Lambda = \bigcup_{\Lambda \in S_n} \Lambda/\mathfrak{m}\Lambda = \{(\Lambda, \xi) : \Lambda \in S_n, \xi \in \Lambda\}$

Revised geometric sorts

$R_{n\ell} = \{(\Lambda, V) : \Lambda \in S_n, V \text{ is an } \ell\text{-dimensional subspace of } \mathrm{res}(\Lambda)\}$

R_{n0} is just S_n, R_{n1} is a projectivised version of T_n

An element (Λ, V) of $R_{n\ell}$ is coded by Λ (an element of R_{n0}) and V thought of as a 1-dimensional subspace of the ℓth exterior power of Λ, which is an element of R_{N1}, where $N = \dim(\wedge^\ell K^n)$.

Lemma

An element of R_{n1} is coded in $\mathcal{S} \cup \mathcal{T}$.

Satisfying the conditions of the EI criterion: 3)

Assume $G = R_{n\ell} = \{(\Lambda, V) : \Lambda \in S_n, \text{V an } \ell\text{-dimensional subspace of } \Lambda\ \}$.

Let $G' = \widetilde{R_{n\ell}} = \{(b, \Lambda, V) : b \text{ is a basis for } \Lambda, (\Lambda, V) \in R_{n\ell}\}$.

Naming a basis b induces a map from V to an ℓ-dimensional subspace W of k^n. The set of such subspaces can itself be embedded in some k^t by elimination of imaginaries in ACF. Thus the map

$$\widetilde{R_{n\ell}} \to K^{n^2} \times k^t \text{ given by } (b, \Lambda, V) \mapsto (b, W) \mapsto (b, \ulcorner W \urcorner)$$

is the required map π.

Comments on proof of Theorem 3

Theorem 3

The code for a definable valued vector space structure on K^m is interdefinable with an element of the geometric sorts.

Recall the construction of a valued field. Field K, valuation ring $\mathcal{O}_K$, value group $\Gamma = K^\times/\mathcal{O}_K^\times$ has total order defined by $x \leq y \iff y/x \in \mathcal{O}_K$. This works because for every pair $x, y \in K$, either $x/y \in \mathcal{O}$ or $y/x \in \mathcal{O}$. The quotient map from $K^\times$ to Γ is the valuation. Mimic this construction on a vector space.

Comments on proof of Theorem 3

Definable valued vector space

Let V be a definable vector space over K, R a definable relation on $V \times V$ such that

$$\Gamma(V) = V \setminus \{0\} / \{(w, v) \in R \,\&\, (v, w) \in R\}$$

has a total order induced by $(w, v) \in R$. Write $\mathrm{val} : V \to \Gamma(V)$ for the quotient map, and suppose that

$$\mathrm{val}(w + v) \geq \min\{\mathrm{val}(w), \mathrm{val}(v)\},$$

and there is an action of Γ induced on $\Gamma(V)$ such that $\mathrm{val}(aw) = \mathrm{val}(a \cdot 1) + \mathrm{val}(w) = v(a) + \mathrm{val}(w)$.

Then the action of Γ on $\Gamma(V)$ has fewer than $\dim(V)$ many orbits.
Theorem 3: $\ulcorner R \urcorner$ is interdefinable with an element of the geometric sorts.

Comments on proof of Theorem 3

$V = K^m$ for some m.
Suppose there is just one orbit in the action of Γ on $\Gamma(V)$.
Then $\ulcorner R \urcorner$ is interdefinable with $B_{\geq 0}(\mathbf{0})$, because $\mathrm{val}(w) \leq \mathrm{val}(u)$ if and only if u is in every closed ball around $\mathbf{0}$ that contains w.
Since $B_{\geq 0}(\mathbf{0})$ is an $\mathcal{O}_K$-submodule of K^m it is coded in the geometric sorts.

More generally, there are at most m orbits. In each of these orbits, find an $\ulcorner R \urcorner$-definable element. Each one defines a closed ball in the valued vector space, which will be an $\mathcal{O}_K$-submodule of K^m, hence coded. The sequence of these codes is interdefinable with $\ulcorner R \urcorner$.

Comments on proof of Lemma 1

Lemma 1

Definable types in $\mathrm{Sym}^n(K^m \times K^\ell)$ have geometric codes.

We already observed that definable types in K^m have geometric codes.
As $\mathcal{O}_K \subset K$, also definable types in $\mathcal{O}_K^\ell$ have geometric codes.
The function $\mathrm{res} : \mathcal{O}_K \to k_K$ has the property that, for every $b \in k_K$ there is a b-definable type in $\mathrm{res}^{-1}(b)$, hence definable types in $\mathcal{K}^m \times k^\ell$ have geometric codes.
Show that a definable type from $\mathrm{Sym}^n(K^m \times K^\ell)$ has the same code as a definable type in $K^{m'} \times k^{\ell'}$, which suffices.

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