Statistical Aspects of Quantum State Monitoring for (and by) Amateurs

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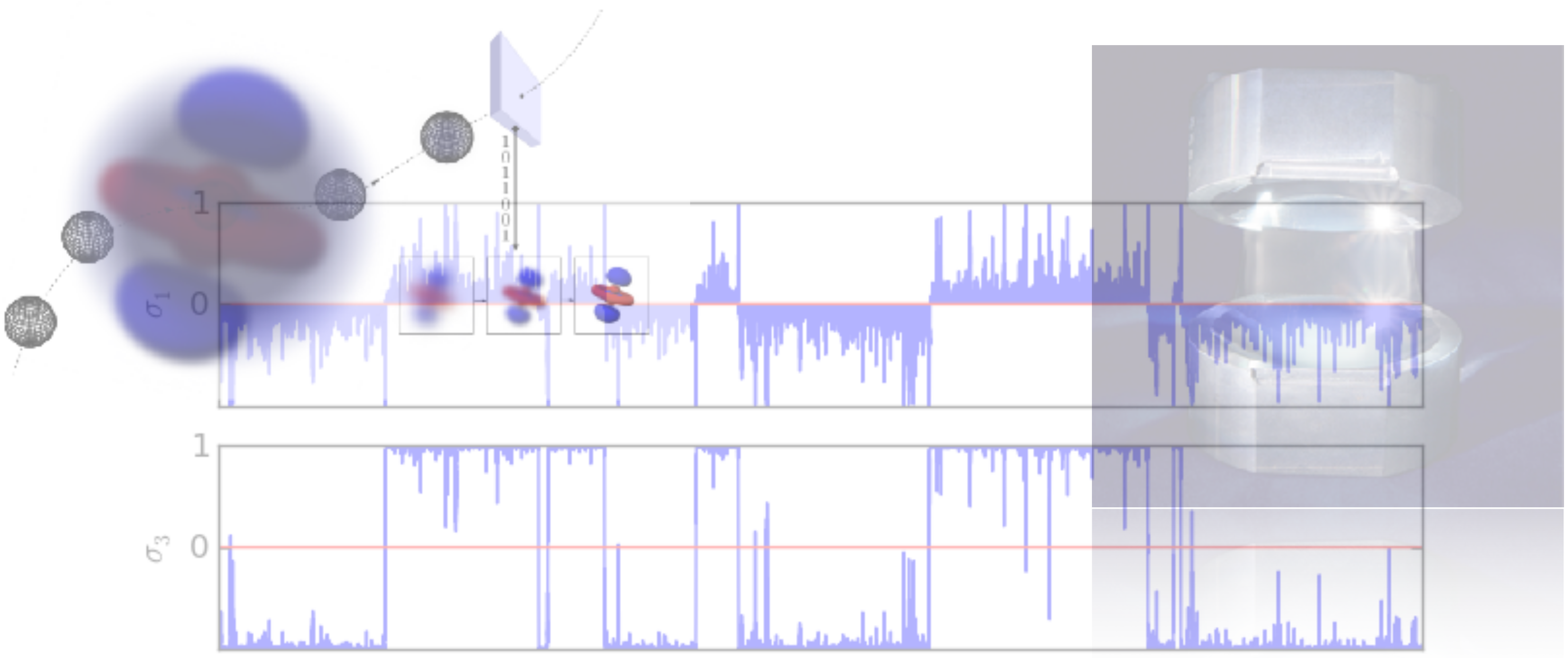


Four lectures:

1- Quantum non-demolition (QND) measurements

2- Discrete quantum trajectories and open quantum walks

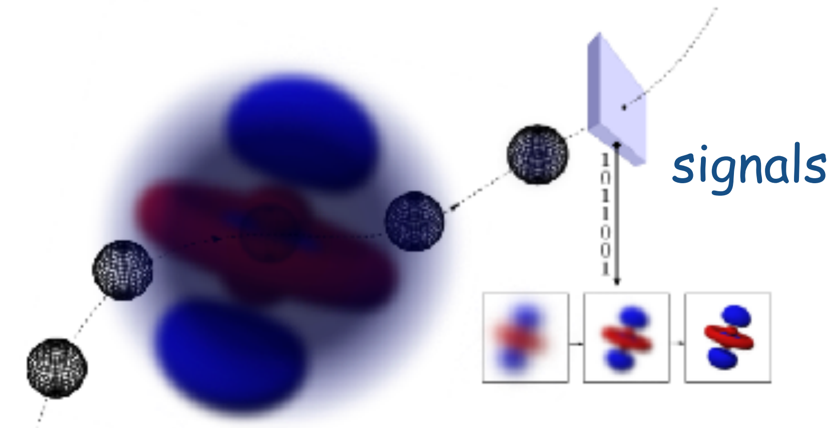
3- Continuous monitoring and quantum trajectories

Scaling limit of POVMs and Lindblad evolution
Quantum trajectory SDEs
Basic examples [on Qu-bits]
Applications

4- Strong monitoring limit

Lecture Notes: https://www.phys.ens.fr/~dbernard/

Recall from lectures 1-2:



— **Iterated POVMs** with output **signal** s_k and **updating**:

$$\rho \to \frac{F_s\,\rho\,F_s^\dagger}{\pi(s)}, \quad \text{with probability } \pi(s) = \mathrm{Tr}(F_s\,\rho\,F_s^\dagger),$$

— To simplify: suppose that probes are spin 1/2, i.e. **a doublet of POVM,**
label the elements of the POVM by $F_\pm$, with $F_+^* F_+ + F_-^* F_- = \mathbb{I}$.

— To take the continuous limit, the F's have to depend on a small parameter and have to be small deformation of trivial elements.
They are **two cases**:

« dispersive »

– either $F_\pm^* F_\pm$ are both non zero and proportional to the identity at $\varepsilon = 0$;
or one of the two $F_\pm^* F_\pm$ vanishes and the other one is equal to the identity at $\varepsilon = 0$.

« click », alias « poisson jumps »

Scaling limit of POVMs and Lindblad evolution (I)

— We aim at taking **the continuous limit** (in time),
with **continuous monitoring and information extraction.**

— We look at the case with the Fs close to the identity, symmetric at e=0.
They have to solve $F_+^* F_+ + F_-^* F_- = \mathbb{I}.$

— The `**local-infinitesimal' form of the POVM** is:

$$F_\pm = \frac{1}{\sqrt{2}} \left[\mathbb{I} \pm \sqrt{\varepsilon} N - \varepsilon(iH \pm M + \frac{1}{2} N^\dagger N) + O(\varepsilon^{3/2})\right]$$

with **H hermitic but arbitrary N and M** (not necessarily hermitic)

This codes for short time (rescaled) interactions between the system and the probe.

-> Check this is compatible the POVM condition…

Scaling limit of POVMs and Lindblad evolution (II)

— Let us first look at the mean behaviour (no records o the outputs +/-):
The transformation of the **mean system state** is:

$$\bar\rho \to \Phi(\bar\rho) = F_+\bar\rho F_+^\dagger + F_-\bar\rho F_-^\dagger$$

— In the infinitesimal form:

$$\delta\bar\rho := \Phi(\bar\rho) - \bar\rho = \big(-i[H,\bar\rho] + L_N(\bar\rho)\ \big)\epsilon$$

with $L_N(\rho) = N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N).$ a « Lindbladian »

Identifying $\epsilon = dt$ gives « **Lindblad evolution** » [one parameter family of continuous CP-maps]

$$d\bar\rho := \big(-i[H,\bar\rho] + L_N(\bar\rho)\ \big)dt$$

Scaling limit of POVMs and Lindblad evolution (III)

Definition-Proposition (Lindblad, ...) "Lindladian":
Let $\mathcal{H}$ be a Hilbert space and let ρ denote quantum states on $\mathcal{H}$. Let H be hermitian and L_a be set of bounded operators on $\mathcal{H}$. Lindblad operators are linear maps on quantum states defined by

$$L(\rho) = -i[H,\rho] + \sum_a [L_a \rho L_a^\dagger - \frac{1}{2}(L_a^\dagger L_a \rho + \rho L_a^\dagger L_a)].$$

Lindblad operators are generators of CP-maps in the sense that the maps $\Phi_t := e^{tL}$ form a one parameter group of CP-maps.
– Reciprocally, any one parameter group of CP-maps, depending continuously on this parameter, can be written in such exponential form.

— More generally/Alternatively:

(Fs) a POVM s.t. $\hat{F}_s = u_s \left(\mathbb{I} + \sqrt{\epsilon}\, N_s + \epsilon\, M_s + \cdots \right) = u_s\, F_s$

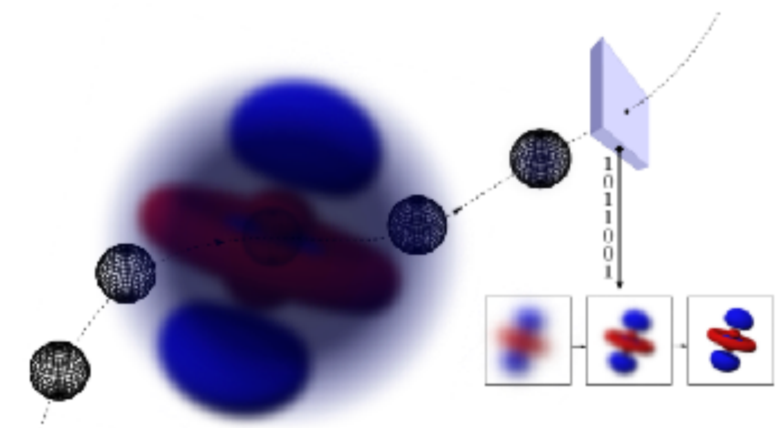
Then, $p_s = |u_s|^2, \quad \sum_s p_s = 1,$ a probability measure $\mathbb{E}^0$

CP-maps: $\rho \to \Phi(\rho) = \mathbb{E}^0[\, F \rho F^\dagger \,]$

—> What the condition for the small epsilon to exist?
What is then the Lindbladian?

$$\Phi(\rho) = \rho + \epsilon\, \mathcal{L}(\rho) + \cdots$$

Scaling limit of POVMs & Quantum trajectory SDEs



— What is the system state evolution
if we keep track of the output signal?
It has to be **random**.

— Recall the one-step transformation (with output +/-)

$$\rho_n \to \rho_{n+1} = \frac{F_\pm\, \rho_n\, F_\pm^\dagger}{\pi_n(\pm)}, \quad \text{with proba } \pi_n(\pm) = \mathrm{Tr}(F_\pm\, \rho_n\, F_\pm^*).$$

— After series of n-steps, we get the **signal** (s1,s2,…sn) = (+,-,-,+,….,-,+),
in 1-to-1 correspondence with random walk.
Its scaling limit is naturally linked to a Brownian motion.
(but the statistic is not going to be that of Brownian motion).

— Let us codes **the signal** in the rescaled sum: $X_n = \sqrt{\varepsilon}\, \sum_{k\leq n} s_k$.

Cf. correspondance with open quantum walks
(and the relation between random walk and Brownian motion)

Scaling limit of POVMs & Quantum trajectory SDEs

— Let us codes **the signal** in the rescaled sum: $X_n = \sqrt{\varepsilon}\,\sum_{k\leq n} s_k.$

In the scaling limit (X_n, ρ_n) becomes time dependent functions (X_t, ρ_t)

— **The scaling limit (** $n \to \infty$ with $t = n\varepsilon$ fixed **) of repeated POVMs** the time evolution is governed by the following SDEs

$$\left|\begin{array}{rcl} d\rho_t &=& -i[H,\rho_t]\,dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t, \\ dX_t &=& \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\,dt + dB_t. \end{array}\right.$$

with

$$\left|\begin{array}{rcl} L_N(\rho) &=& N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N), \\ D_N(\rho) &=& N\rho + \rho N^\dagger - \rho\,\mathrm{Tr}(N\rho + \rho N^\dagger) \end{array}\right.$$

The measured observable is N+N*. And Bt is a Brownian motion.
Alternative representation in terms of the signals. $X_t \equiv S_t \equiv \text{"signal"}$

— More generally: « (diffusive) **Quantum Trajectory SDEs** »

$$d\rho_t = \Big(-i[H,\rho_t] + \sum_a L_{M_a}(\rho_t)\Big)dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t$$

and with more Brownian (or signals) sources....

Quantum trajectory SDEs: relation with Filtering

Cf. John's lectures.

— Quantum trajectory equations are non-linear.

$$\begin{cases} d\rho_t &= -i[H,\rho_t]\,dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t, \\ dX_t &= \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\,dt + dB_t. \end{cases}$$

— But… solution can be written as: $\rho_t = \sigma_t / Z_t \quad \text{with} \quad Z_t = \mathrm{Tr}(\sigma_t)$

i.e. \sigma is an « unnormalised state ».

with

$$\begin{cases} d\sigma_t = i[H,\sigma_t]dt + L_N(\sigma_t) + (N\sigma_t + \sigma_t N^\dagger)\,dX_t \\ dZ_t = Z_t\,\mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\,dX_t \end{cases}$$

This is now a linear equation in sigma, driven by the output signal.
Compare with Filtering equation….
Compare with the iteration of the discrete map: $\rho \to \dfrac{F_s \rho F_s^\dagger}{\pi(s)}$

Quantum trajectory SDEs: Hint for a proof:

— Let the signal be: $X_n = \sqrt{\varepsilon}\, \sum_{k \le n} s_k$.

$$\begin{aligned} \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] &= \sqrt{\varepsilon}\, \mathbb{E}[s_{n+1} | \mathcal{F}_n] = \sqrt{\varepsilon}\, (\pi_n(+) - \pi_n(-)) = \varepsilon\, \mathrm{Tr}(N\rho_n + \rho_n N^\dagger) + \cdots . \\ \mathbb{E}[(X_{n+1} - X_n)^2 | \mathcal{F}_n] &= \varepsilon\, \mathbb{E}[s_{n+1}^2 | \mathcal{F}_n] = \varepsilon . \end{aligned}$$

There is a drift in the statistics because the probability to have +/- is state dependent:

This corresponds to: $dX_t = \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\, dt + dB_t$.

— Iteration of the discrete Q-trajectories

$$\rho_1 = \frac{F_{s_1} \rho_0 F_{s_1}^\dagger}{\pi(s_1)} \quad , \qquad \rho_2 = \frac{F_{s_2} \rho_1 F_{s_2}^\dagger}{\pi(s_2)} = \frac{F_{s_2} F_{s_1} \rho_0 F_{s_1}^\dagger F_{s_2}^\dagger}{\pi(s_1, s_2)} \qquad \text{, etc....}$$

Thus $\rho_n = \dfrac{\sigma_n}{Z_n}$ with $\sigma_n = F_{s_n} \cdots F_{s_1} \rho_0 F_{s_1}^\dagger \cdots F_{s_n}^\dagger$

Expand in epsilon and compare sigma_{n+m} and \sigma_n

$$d\sigma_t = i[H, \sigma_t]dt + L_N(\sigma_t) + (N\sigma_t + \sigma_t N^\dagger)\, dX_t$$

Unraveling of Lindblad equation

— Represent a dissipative Lindblad equation has the mean expectation of the non-dissipative/non-mixing but random process.

$$d\rho_t = -i[H, \rho_t]\, dt + L_N(\rho_t)\, dt$$

Cf. the discrete case, or quantum noise.

— Diffusive Quantum trajectories as unraveling... (case i) (see above)

$$d\rho_t = L_{\mathrm{sys}}(\rho_t)\, dt + L_N(\rho_t)\, dt + D_N(\rho_t)\, dB_t$$

— Jumpy Poisonnian Q-trajectories as unraveling... (case ii)

$$d\rho_t = L_{\mathrm{sys}}(\rho_t)\, dt + L_N(\rho_t)\, dt + \hat{M}_N(\rho_t)\, dY_t$$

with $\hat{M}_N(\rho) = \frac{N\rho N^\dagger)}{\mathrm{Tr}(N\rho N^\dagger)} - \rho$ and $dY_t = dN_t - \mathrm{Tr}(N\rho N^\dagger)\, dt$

Poisson process with intensity/mean $\mathrm{Tr}(N\rho N^\dagger)$

—> Typical trajectory during time interval dt?

Lindblad evolution and (Poisson) unraveling

— Following the approach we adopted, we got the Lindblad equations via the mean evolution of random POVMs.

—> We can also view them as the mean evolution of « jumpy » processes.

— Start from $d\rho_t = -i[H,\rho_t]\,dt + L_N(\rho_t)\,dt$

— Define a random evolution (on states) between t and t+dt via:

- If no jumps: $|\psi_t\rangle \to |\psi_t\rangle - iH\,|\psi_t\rangle\,dt - \frac{1}{2}\big(N^\dagger N - \langle N^\dagger N\rangle_t\big)\,|\psi_t\rangle\,dt$

  with proba $\Big(1 - \langle N^\dagger N\rangle_t\,dt\Big)$

- If jumps: $|\psi_t\rangle \to N|\psi_t\rangle/\sqrt{\langle N^\dagger N\rangle_t}$

  with proba $\langle N^\dagger N\rangle_t\,dt$

—> Cf. Dyson's expansion for the kernel « exp(tL) »…

Bacis examples of Quantum Trajectories

Qu-bit
Spin 1/2

$$\rho = \begin{pmatrix} Q & U \\ U^* & 1-Q \end{pmatrix}$$

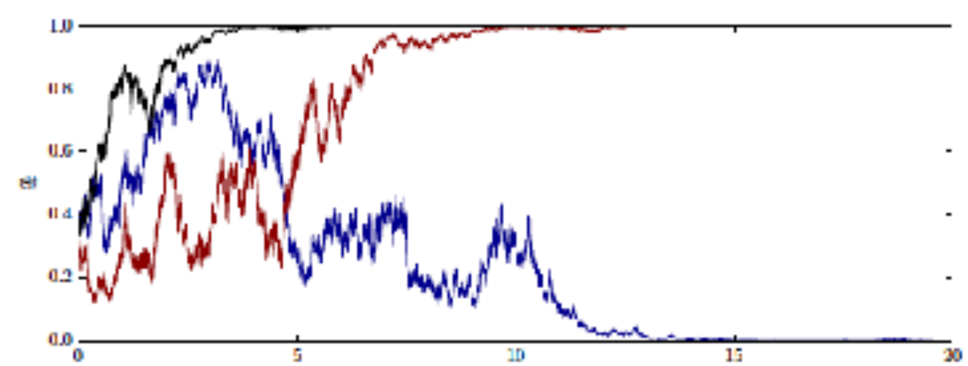
— **Example I: Non-demolition measurement**

The continuous version of the discrete **non-demolition measurement**.

Q-trajectory SDEs: $$d\rho = L_{\rm meas}(\rho)\, dt + D_{\rm meas}(\rho)\, dB_t$$

If sigma^z is monitored: $$L_{\rm meas}(\rho) = -\frac{\gamma^2}{32}[\sigma_z, [\sigma_z, \rho]] \quad \text{and} \quad D_{\rm meas}(\rho) = \frac{\gamma}{4}(\{\sigma_z, \rho\} - 2\rho\, \mathrm{tr}(\rho\sigma_z)).$$

$Q_t = {}_z\langle +|\rho_t|+\rangle_z$



— SDEs for the diagonal matrix element Q:

$$dQ_t = \gamma\, Q_t(1 - Q_t)\, dB_t,$$

Hence, Qt is a (bounded) martingale.
The martingale **convergence theorem** applies an

$$Q_\infty = 1 \text{ with proba } Q_0, \quad Q_\infty = 0 \text{ with proba } 1 - Q_0.$$

The convergence is exponentially fast with a time scale of order γ^{-2}.

— Find an explicit solution to this equation (in terms of the signals)?

— Example II: Qu-bit in thermal contact

A Qu-bit, in contact with a thermal bath, **with its energy continuously monitored**.

$$d\rho = (d\rho)_{\mathrm{syst}} + (d\rho)_{\mathrm{meas}}. \qquad \text{with} \qquad (d\rho)_{\mathrm{syst}} = -i[h,\rho]dt + L_{\mathrm{therm}}(\rho)\,dt$$

$$L_{\mathrm{therm}}(\rho) = \lambda p(\sigma \ \rho\sigma_{|} - \frac{1}{2}\{\sigma_{|}\sigma \ ,\rho\}) + \lambda(1-p)(\sigma_{|}\rho\sigma \ - \frac{1}{2}\{\sigma \ \sigma_{|},\rho\}),$$

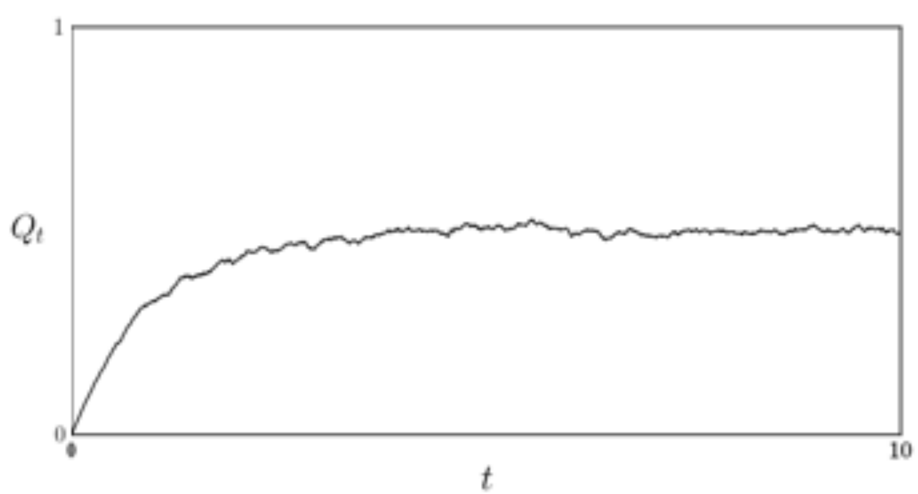
(induce incoherent transitions between states |+> and |->)

— SDEs for the diagonal matrix element Q:

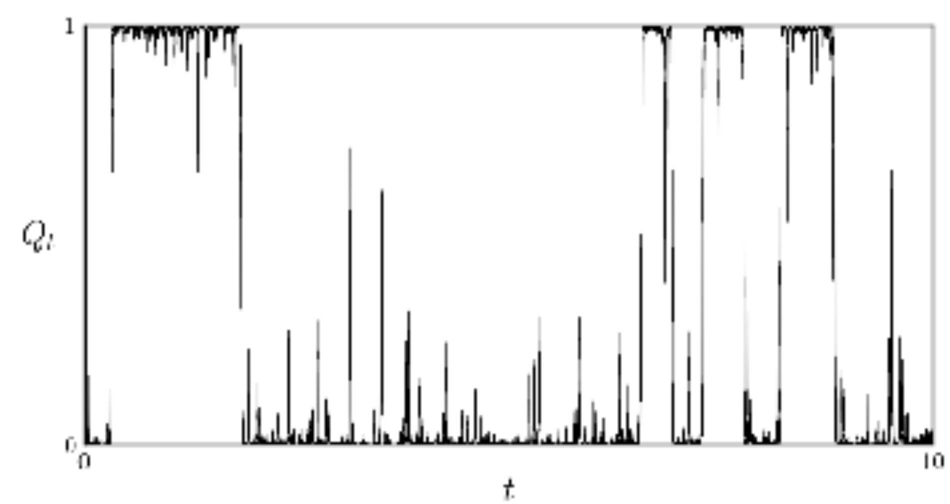
$$dQ_t = \lambda(p - Q)dt + \gamma\, Q_t(1 - Q_t)\, dB_t.$$

Thermal relaxation **Effect of monitoring**

« competition »
with two characteristic time scales!



large lambda



large gamma

— Example III: Monitored Rabi oscillation

A Qu-bit with the monitoring of an observable **non-commuting** with the hamiltonian.

Here we monitor the system observable σ^z for a Qu-bit with hamiltonian $\omega\sigma^x$.

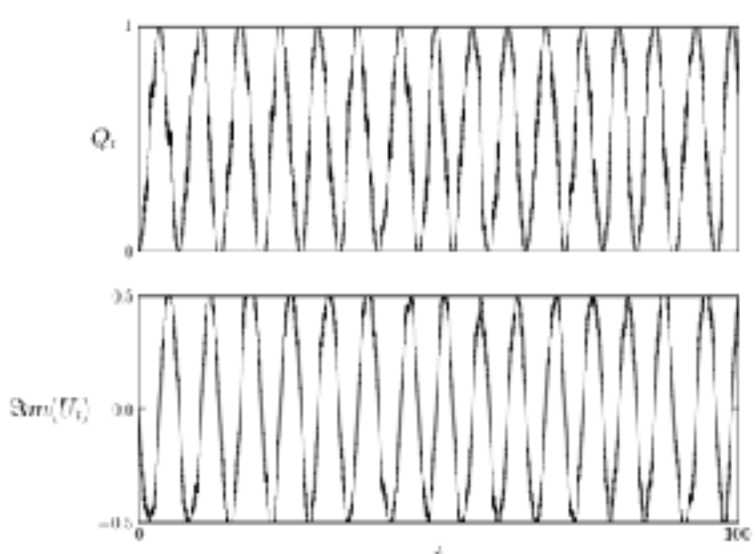
— **Q-trajectory SDEs** $d\rho = (d\rho)_{\rm sys} + (d\rho)_{\rm meas}$ with $(d\rho)_{\rm sys} = -i\omega[\sigma^x, \rho_t]\, dt$

$$\begin{cases} dQ_t = \omega\, U_t\, dt + \gamma Q_t(1-Q_t)\, dB_t, \\ dU_t = -\omega\,(Q_t - \frac{1}{2})\, dt - \frac{\gamma^2}{8} U_t\, dt - \gamma U_t(Q_t - \frac{1}{2})\, dB_t \end{cases}$$

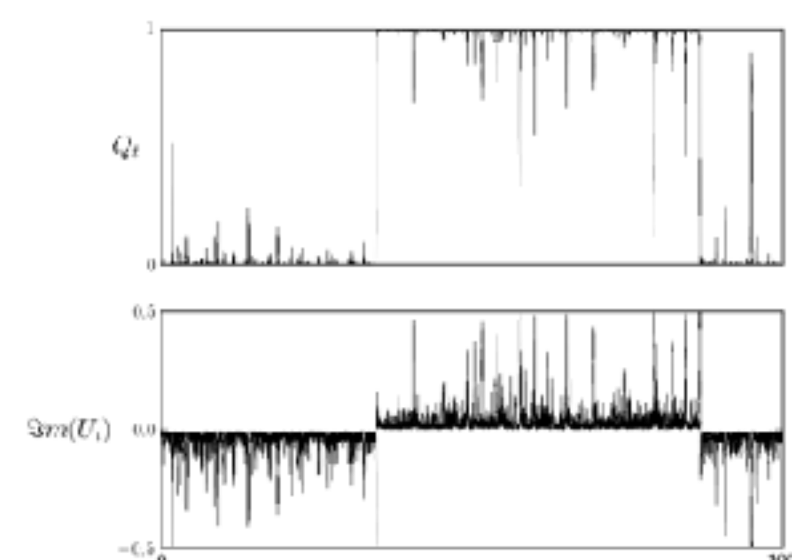
for $\rho = \begin{pmatrix} Q & U \\ U^* & 1-Q \end{pmatrix}$

Rabi oscillation ↗ « **competition** » ↖ Effect of measurement

with two characteristic time scales!



large omega



large gamma

— The state purifies exponentially fast: reduction to a one-parameter SDE

Application (I): The Open Quantum Brownian Motion (OQBM)

— **OQBM = Scaling limit of OQW.**

— Recall that OQW = repeated position dependent POVM (alias Q-trajectories)

if $\sum_x \bar\rho_n(x) \otimes |x\rangle\langle x| := \mathbb{E}[\rho_n \otimes |x_n\rangle\langle x_n|]$ is the mean density matrix of an OQW

then $\bar\rho_{n+1}(x) = \mathfrak{P}(\bar\rho_n)(x) = \sum_y B_{yx}\bar\rho_n(y)B^*_{yx},$ with B's the transition matrices

— **For an homogeneous walk on the line:**

$$\rho_{t+\delta t}(x) = F_-\,\rho_t(x+\delta x)\,F_-^\dagger + F_+\,\rho_t(x-\delta x)\,F_+^\dagger$$

— In the **scaling limit**, with diffusive scaling (dx^2=dt) and $F_\pm = \frac{1}{\sqrt{2}}\,[\mathbb{I} \pm \sqrt{\varepsilon}N - \varepsilon(iH + \frac{1}{2}N^\dagger N) + O(\varepsilon^{3/2})]$

$$\partial_t\bar\rho_t(x) = -i[H,\bar\rho_t(x)] + \frac{1}{2}\partial_x^2\bar\rho_t(x) - \left(N\partial_x\bar\rho_t(x) + \partial_x\bar\rho_t(x)N^\dagger\right) + L_N(\bar\rho_t(x)).$$

This is a (well defined) Lindblad equation on H x L^2(R), called the OQBM, generalizing diffusion equations, mixing spatial and internal d.o.f.'s.

— Can be generalized to **higher dimension with/without in-homogeneities.**

Application(II): Elements of feedback & control

See I. Dotsenko's and B. Huard lectures

— System monitoring -> information/output signals
-> back acts on the systems
-> many ways (at least theoretically…)

— Say:

- monitoring : $d\rho_t = -i[H, \rho_t]\, dt + L_N(\rho_t)dt + D_N(\rho_t)dB_t,$
- signals: $dX_t = \mathrm{Tr}(N\rho_t + \rho_t N^\dagger)\, dt + dB_t.$
- back-act unitarily: $\rho \to e^{-i\mathfrak{h}\, dX_t}\, \rho\, e^{+i\mathfrak{h}\, dX_t} = \rho - i[\mathfrak{h}, \rho]\, dX_t - \frac{1}{2}[\mathfrak{h}, [\mathfrak{h}, \rho]]\, (dX_t)^2$

—> What is the total evolution ?

— Many other protocols:

Four lectures:

1- Quantum non-demolition (QND) measurements

2- Discrete quantum trajectories and open quantum walks

3- Continuous monitoring and quantum trajectories

4- Strong monitoring limit

Examples: Thermal Qu-bit & Rabi
The quantum jump Markov chain
Examples
A finer structure: quantum spikes

Two basics examples:

—**Two processes in competition**: system evolution and monitoring with different time scales.

A thermal Qu-bit:

Monitoring the energy of Qu-bit in contact with a thermal bath

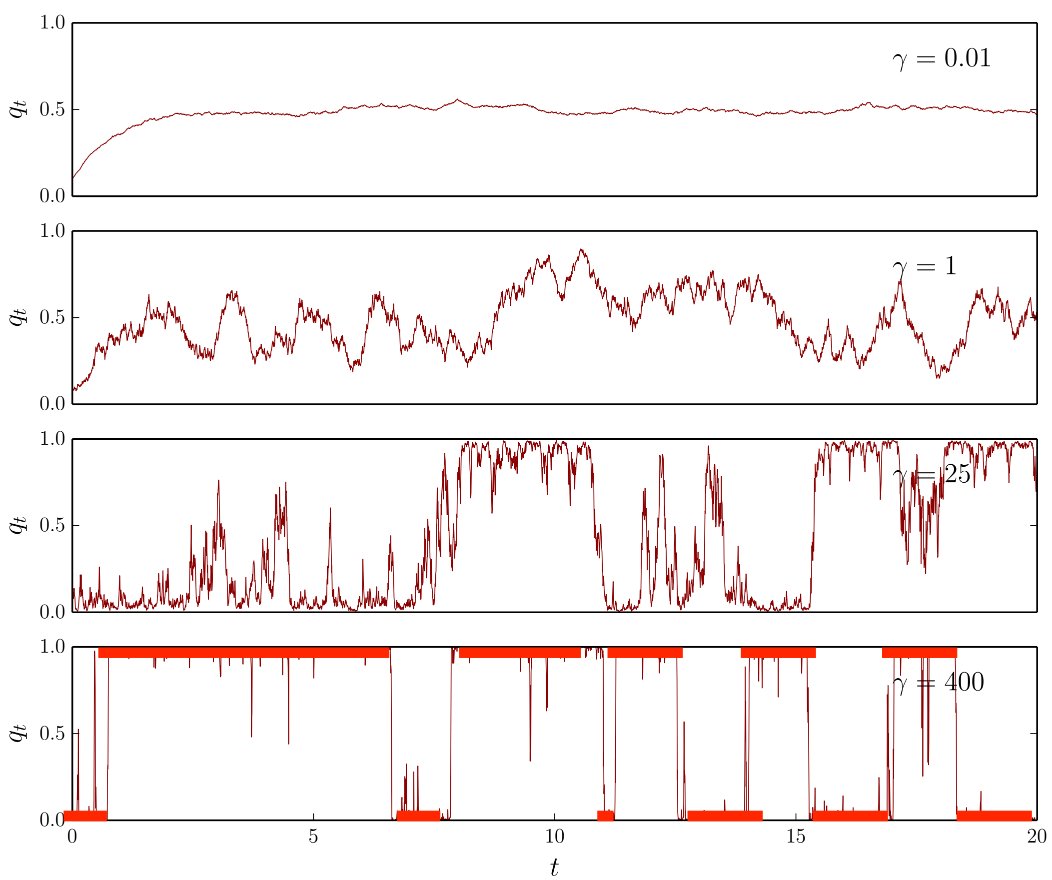
$\mathcal{O} = H \propto \sigma_z$

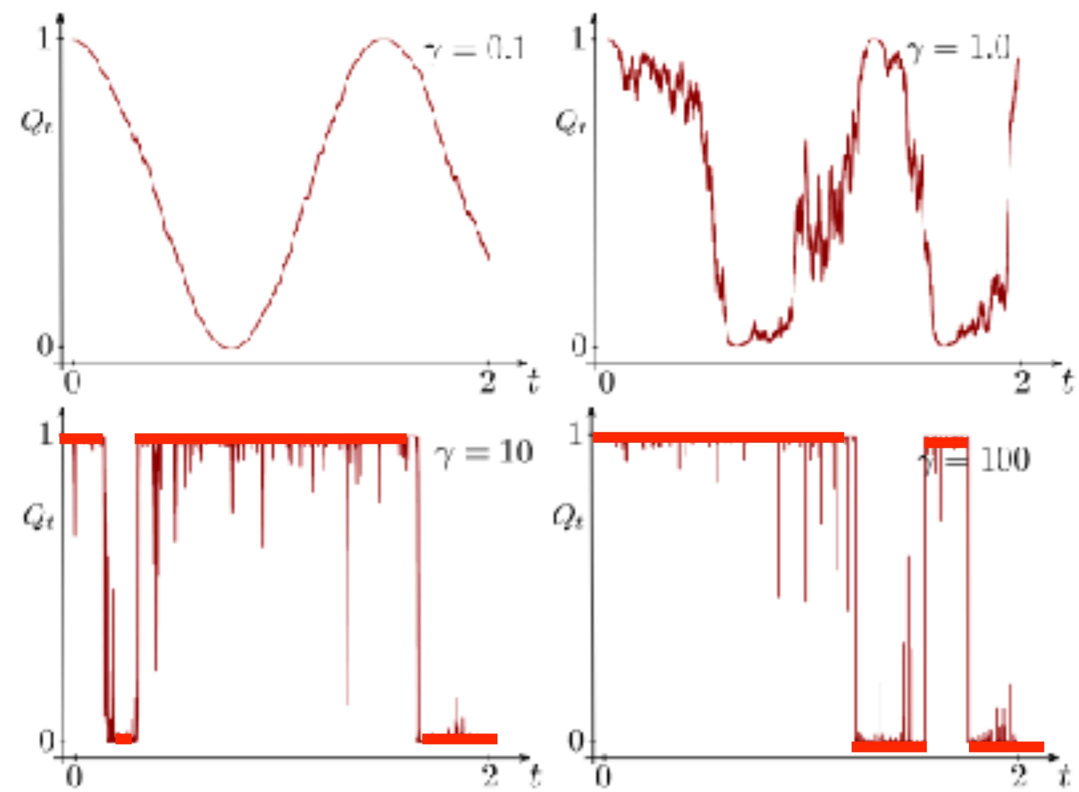
A coherent Qu-bit:

Monitoring Rabi oscillation

$H = \frac{\Omega}{2}\sigma_y \qquad \mathcal{O} = \sigma_z$

$Q_t = {}_z\langle + | \rho_t | + \rangle_z$





(As T-meas decreases)

— Quantum jumps emerge (from a diffusive behaviour, they are not built in)… because we moved the `von Neumann cut'.

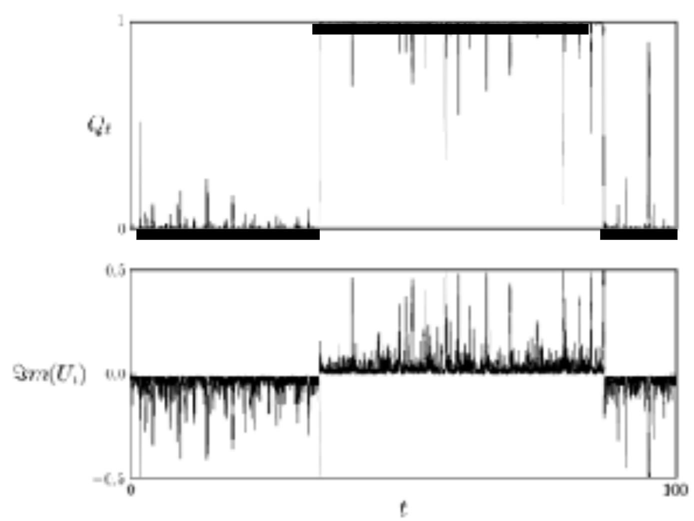
Zeno effect or not?

— How does the **quantum Zeno effect** affects quantum trajectories ?

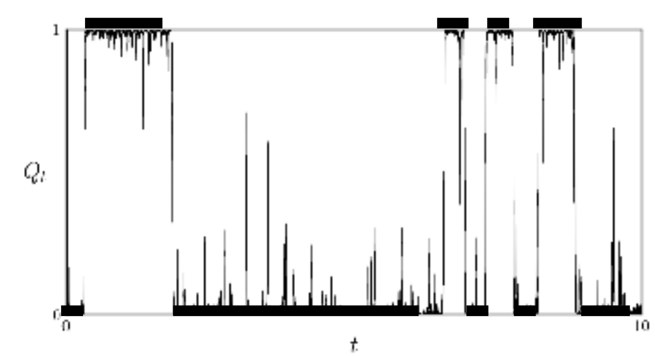
— Recall how do quantum trajectories for a thermal Qu-bit or for Rabi oscillations look.

$$dQ_t = \omega\, U_t\, dt + \gamma Q_t(1-Q_t)\, dB_t,$$
$$dU_t = -\omega\,(Q_t - \frac{1}{2})\, dt - \frac{\gamma^2}{8} U_t\, dt - \gamma U_t (Q_t - \frac{1}{2})\, dB_t$$

$$dQ_t = \lambda(p-Q)dt + \gamma\, Q_t(1-Q_t)\, dB_t.$$



similar
<- but ->



$$\bar{T}_{\rm jump} = \gamma^2/4\omega^2 \propto \tau^2_{\rm Rabi}/\tau_{\rm meas}.$$

Zeno effect

$$\lim_{\gamma\to\infty} T_\downarrow = 1/\lambda(1-p) \;\; and \;\; \lim_{\gamma\to\infty} T_\uparrow = 1/\lambda p.$$

No Zeno effect

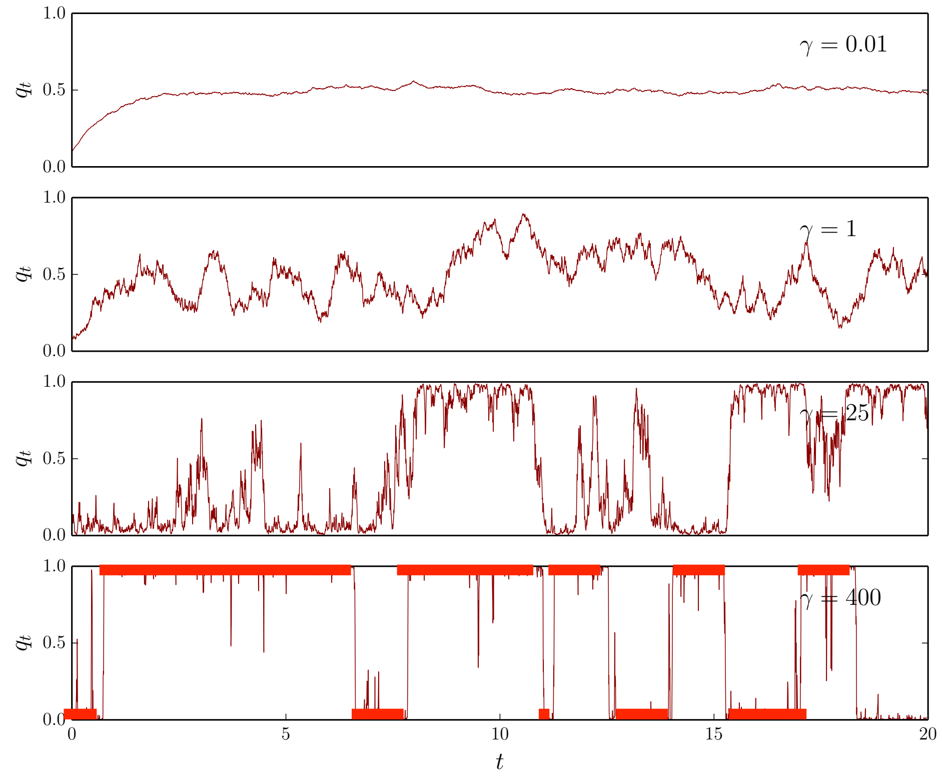
— Hamiltonian and dissipative channels react quite differently to measurement back action, i.e. to the Zeno effect [because of the number of d.o.f.'s involved]

Basic example: Thermal Qu-bit

or... **what is the strong measurement limit of quantum trajectories (part I)**

— A Qu-bit, in contact with a thermal bath with its energy continuously monitored.

$$dQ_t = \lambda(p - Q)dt + \gamma\, Q_t(1 - Q_t)\, dB_t.$$



In the limit $\gamma \to \infty$ of very efficient monitoring, we have :

— *The invariant measure of the thermal quantum trajectory SDE has a limit:*

$$\lim_{\gamma\to\infty} dP_{\rm stat} = [(1-p)\delta(Q) + p\delta(1-Q)]\, dQ.$$

— *The limits of the mean time $T_\downarrow$ (resp. $T_\uparrow$) the trajectories spend near $Q \simeq 1$ (resp. $Q \simeq 0$) are:*

$$\lim_{\gamma\to\infty} T_\downarrow = 1/\lambda(1-p) \ \textit{and}\ \lim_{\gamma\to\infty} T_\uparrow = 1/\lambda p.$$

Basic example: Thermal Qu-bit   **Hint for a proof**

— Stationary measure for 1d SDE: $dQ_t = f(Q_t)dt + g(Q_t)dB_t$

—> Transition kernels, evolution of expectation values, of measures.

Evolution of measure: $dP(Q) = \partial_Q\Big(\frac{1}{2}\partial_Q\, g(Q) - f(Q)\Big)P(Q)\, dt$

Stationary measure: $dP_{\text{stat}}(Q) = 0$

— Standard formula for the stopping time statistics:

Let $0 < Q_i < Q_f < 1$. Let $T_{i\to f}$ be the first instance the process started at Q_i hits Q_f before hitting 0.

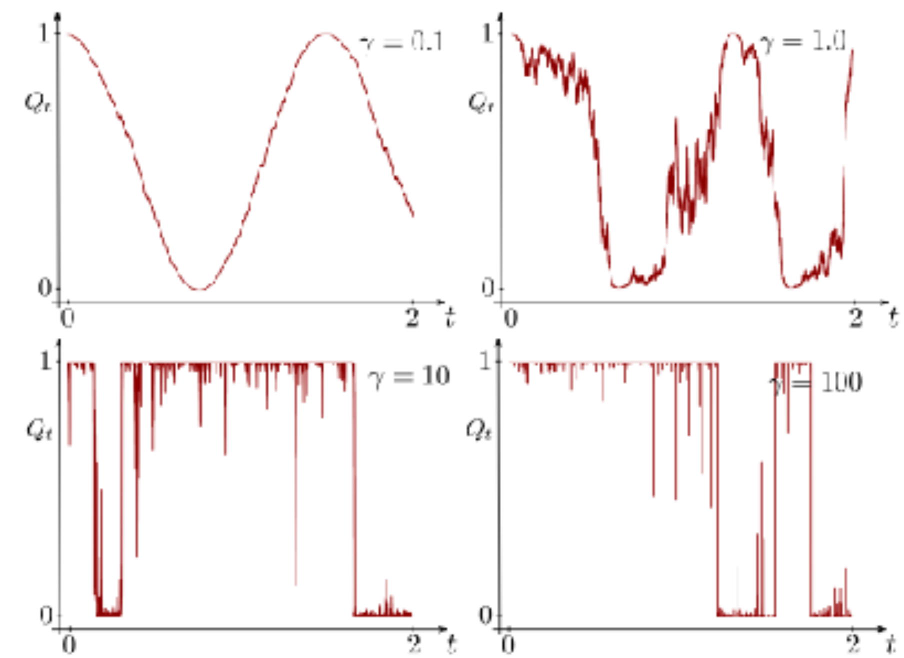
$\mathbb{E}[T_{i\to f}] = 2\int_{Q_i}^{Q_f} dQ\, e^{2h(Q)} \int_0^Q dP_{\text{stat}}.$ with $\partial h(Q) = -f(Q)/g^2(Q)$ [if 0 is a forbidden target]

Then, careful limit gamma->infinity.

— Actually: $\lim_{\gamma\to\infty} \mathbb{P}[\lambda T_{i\to f} \in B] = \frac{Q_i}{Q_f}\,\mathbb{1}_{0\in B} + \Big(1 - \frac{Q_i}{Q_f}\Big)\frac{p}{Q_f}\int_B e^{-s\frac{p}{Q_f}}\, ds.$ [to be used later.....]

Basic example: Monitored Rabi oscill. or... what is the strong measurement limit of quantum trajectories (part I)

— **Similar methods** apply to the case of **monitored Rabi oscillation for a Qu-bit**, to prove the Zeno formula for for the mean time in between jumps $\bar{T}_{\text{jump}} = \gamma^2/4\omega^2$ and finer statistical properties.



— Hint of a proof:

Project the Q-trajectory SDEs on pure state condition so we restrict to states of the form $|\psi_t\rangle = \cos(\theta_t/2)|\uparrow\rangle + \sin(\theta_t/2)|\downarrow\rangle$, which correspond to $Q_t = \frac{1}{2}(1+\cos\theta_t)$ and $U_t = \frac{1}{2}\sin\theta_t$. This reduces the evolution equation to a single SDE for Q_t or θ_t:

$$d\theta_t = -(\omega + 2\gamma \sin\theta_t \cos\theta_t)\, dt - 2\gamma \sin\theta_t\, dB_t.$$

The quantum jumps Markov chain

— **In general**: $d\rho_t = L_{\rm sys}(\rho_t)\,dt + \gamma^2 L_N(\rho_t)\,dt + \gamma\, D_N(\rho_t)\,dB_t.$

To take the large gamma limit, avoiding the Zeno effect requires rescaling coefficients in the system dynamics.

— **First, the mean behaviour**:

Let N be diagonalisable and |i> the eigen-vectors.

Let Qi be the diagonal element of the density matrix. $Q_i := \langle i|\rho|i\rangle.$

Then, $$\partial_t \overline{Q}_j = \sum_i \overline{Q}_i\, m^i_j.$$ **Markov chain on pointer states.**

Explicitly computable from the microscopic data.

If one is only interested in the **mean behaviour** (and hence in the jump rates)

$$d\rho_t = \Big(L_{\rm sys}(\rho_t) + \gamma^2\, L_N(\rho_t) \Big)\, dt$$

The quantum jumps Markov chain (I) : **Hint for a proof.**

— If one is only interested in the **mean behaviour** (and hence in the jump rates)

$$d\rho_t = \Big(L_{\rm sys}(\rho_t) \; + \gamma^2 \, L_N(\rho_t) \Big) \, dt$$

Potentially big at large gamma!!

Perturbation theory around « Ker Ln ».

Two simple cases:

(i) (**Dissipative channel**) L_sys preserves the pointer basis: $L_{\rm sys}(|i\rangle\langle i|) = \sum_j A_{ij} \, |j\rangle\langle j|$

At large gamma: $d\bar\rho_t|_{diag} = L_{sys}(\bar\rho_t|_{diag})dt$

(ii) (**Unitary channel**) : $L_{sys}(\rho) = -i[H,\rho]$ (+ conditions)

At large gamma: $d\rho_t|_{diag} = -\gamma^{-2} \left(L_{sys} \cdot (L_N^{\perp})^{-1} \cdot L_{sys} \right) (\rho_{diag})|_{diag}$

The quantum jumps Markov chain

Potentially big at large gamma!!

— **In general**: $$d\rho_t = L_{\mathrm{sys}}(\rho_t)\,dt + \gamma^2 L_N(\rho_t)\,dt + \gamma\,D_N(\rho_t)\,dB_t.$$

(To take the large gamma limit, avoiding the Zeno effect requires rescaling coefficients in the system dynamics.)

The proof is (we believe) interesting itself because it requires dealing with the **strong noise limit of SDEs**.

— **More:**

> At large gamma, **all finite dimensional distributions** of the conditioned density matrix converge to those of a **finite state Markov process on** the projectors associated to the measurement eigenvectors (**the pointer states**).

This applies only not the finite dimensional distributions.

- At every (fixed) time: $\rho_t = |i_t\rangle\langle i_t|$

- In mean: $\bar{\rho}_t = \sum_i \bar{Q}_i(t)\,|i\rangle\langle i|$

The quantum jumps Markov chain (II) : **Hint for a proof.**

$$d\rho_t = L_{\rm sys}(\rho_t)\,dt + \gamma^2 L_N(\rho_t)\,dt + \gamma\, D_N(\rho_t)\,dB_t.$$

— It is based on an analysis of the second order differential operator associated to the SDEs:

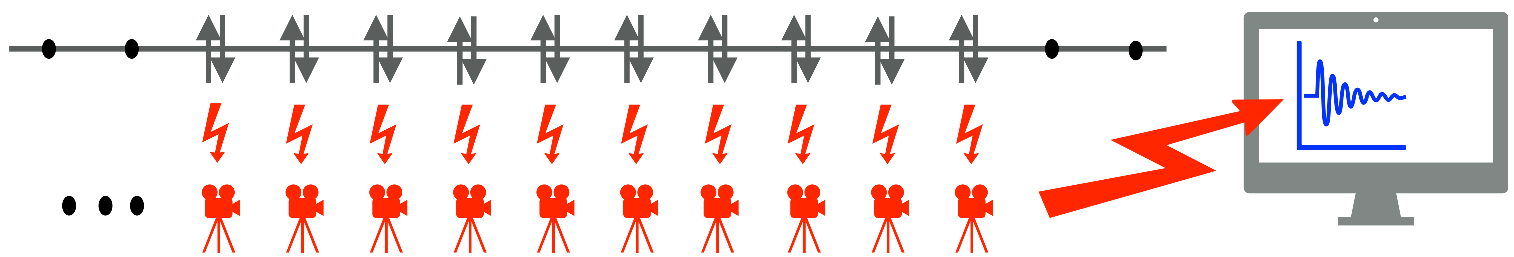
$$df(\rho_t) = (\mathfrak{D} f)(\rho_t)\,dt + (\cdots)dB_t.$$

with D of the form: $\mathfrak{D} = \mathfrak{D}_0 + \gamma^2 \mathfrak{D}_2.$

Projection at large gamma plus perturbation theory.

Potentially big at large gamma!!

Imaging transport & Q-flux in spin chains :



$$d\rho_t = -i[h,\rho_t]\,dt + d\rho_t|_{\text{meas.}}$$

XY Hamiltonian $H = \varepsilon \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)$

with monitoring of the spin Sz : $N_j = \sigma_j^z.$

— Pointer states are spin configurations : $|[\epsilon]\rangle = |\cdots, \uparrow\downarrow, \cdots, \uparrow\downarrow, \cdots\rangle$

—> Projection on (classical) spin configuration (at every time):

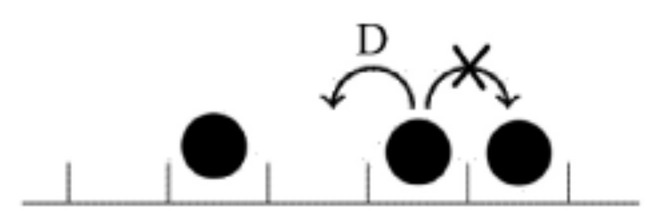
$$\rho_t = |[\epsilon_t]\rangle\langle[\epsilon_t]| = \mathbb{P}(\epsilon_t)$$

but mean state.....

$$\bar{\rho}_s = \sum_\epsilon \bar{Q}_s(\epsilon)\mathbb{P}(\epsilon),$$

$$\frac{d}{ds}\bar{\rho}_s = -\frac{1}{2}D\sum_j \left[\sigma_j^+\sigma_{j+1}^-, [\sigma_j^-\sigma_{j+1}^+, \bar{\rho}_s]\right] + \text{h.c.},$$

(Quantum) Markov chain = SSEP,
alias Simple Symmetric Exclusion Process.



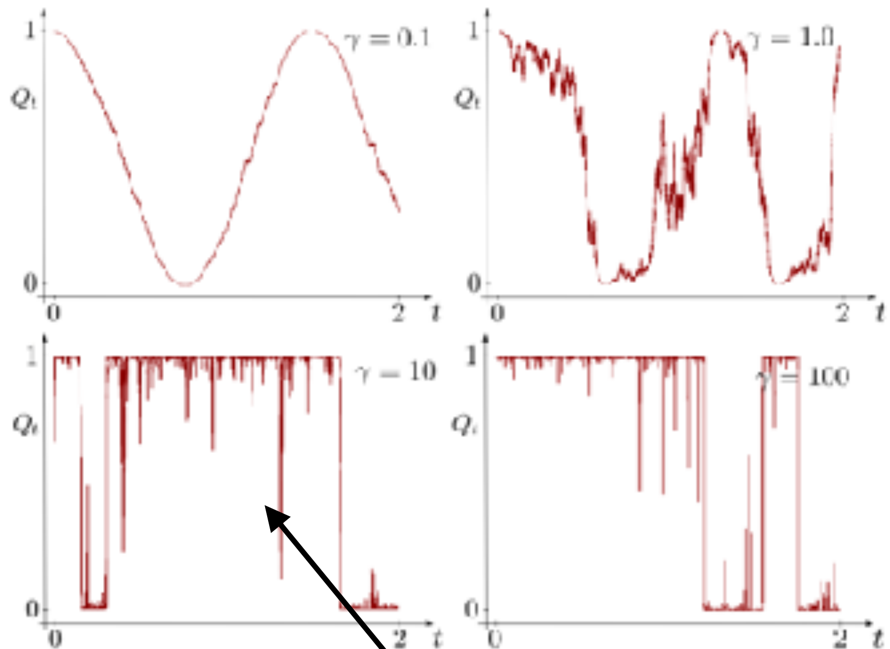
— Emergence of « classicality » from Q-monitoring.

- But classical variables are relative to the monitored observable and the dynamics depends on the monitoring process.

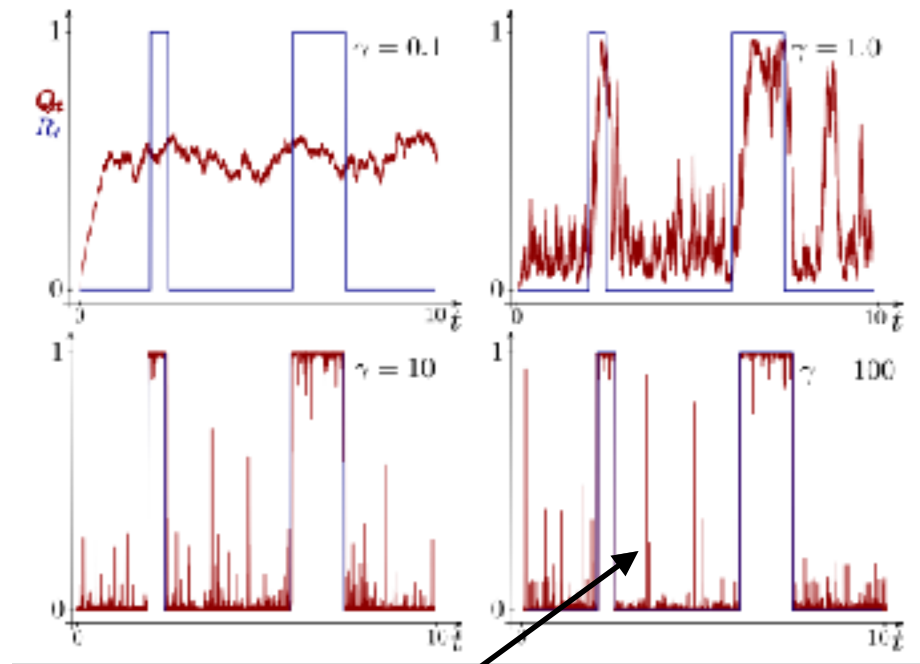
A finer structure: quantum spikes

— Let us look again at the quantum trajectories:

For a coherent Qu-bit:



For a thermal Qu-bit:



A finer structure survives, besides the jumps: the spikes.

- Spikes of height bigger a than a cutoff are countable.
- They have infinitesimal time duration, of order gamma^{-2}

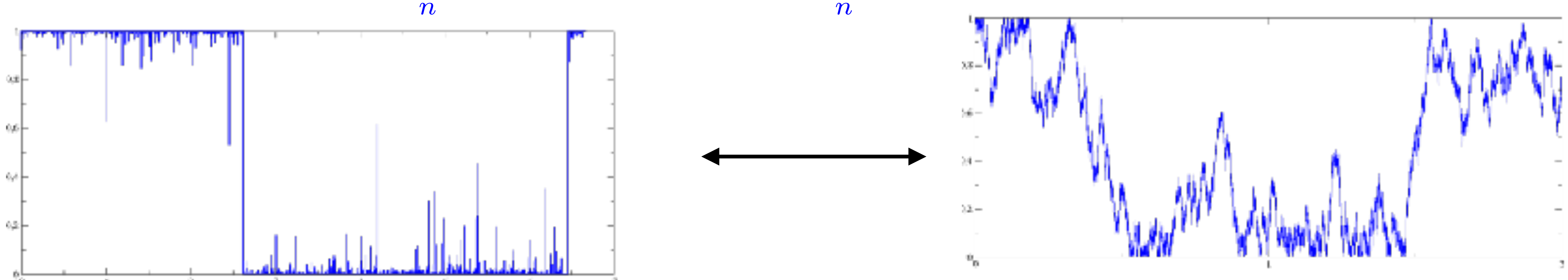
— They have to be taken into account say when controlling Q-bits (as otherwise the `controller' may trigger to often).

Zooming in on Quantum Trajectories

— Q-Spikes are (almost) instantaneous (time scale of order T-meas) but have an internal structure

—> Use a « **effective time** » only sensible to state variation instead of the `natural' time parametrisation. Say:

$$\tau = \sum_n \mathrm{Tr}[(\delta\rho_n)^2)] \quad \text{or} \quad \tau = \sum_n \mathrm{Tr}[(\delta\rho_n)^2)_{\mathrm{diag}}]$$



with natural time parametrisation

parametrised by the quadratic variation

— This is not in contradiction with the fact that all finite distributions of the Q-trajectories converge to those of the jump Markov chain.

« What is true at any (given) time does not (always) hold at every time ! »

Discrete:
system evolution + weak measurement at high frequency.

Continuous:
system evolution + continuous monitoring at high rate.

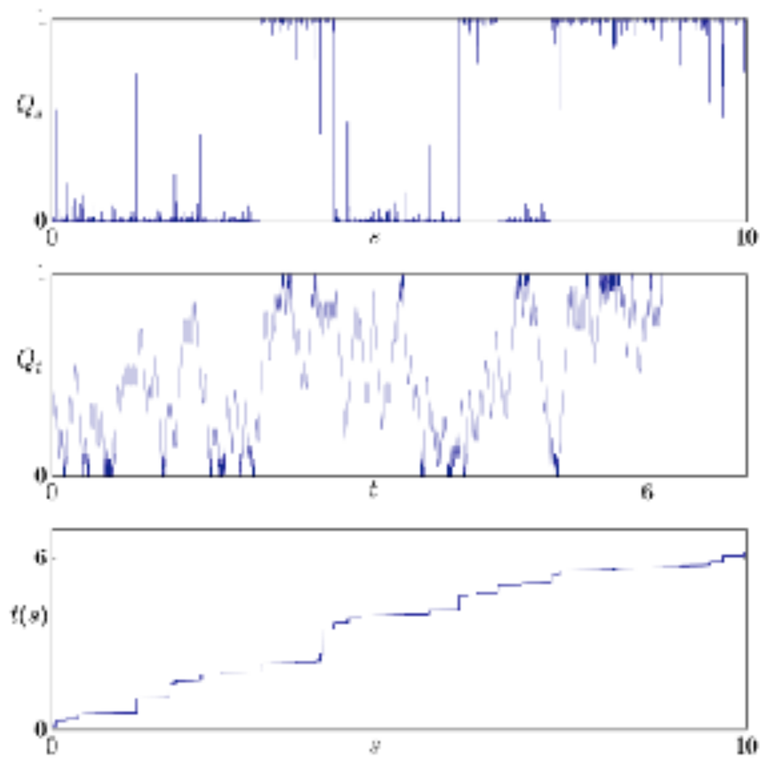


FIG. 1. *Discrete quantum trajectories in real and effective time. Top: The evolution of the ground state probability Q in real time s shows sharp jumps and spikes. Center: The evolution of Q in effective time t allows to resolve what happens outside the boundaries* 0 *and* 1*, the details are unfolded. Bottom: Effective time as a function of the real time. The plots are shown for the same realisation with* $\varepsilon = 0.3$, $\Delta s = 10^{-5}$, $p = 0.5$ *and* $\lambda = 1$.

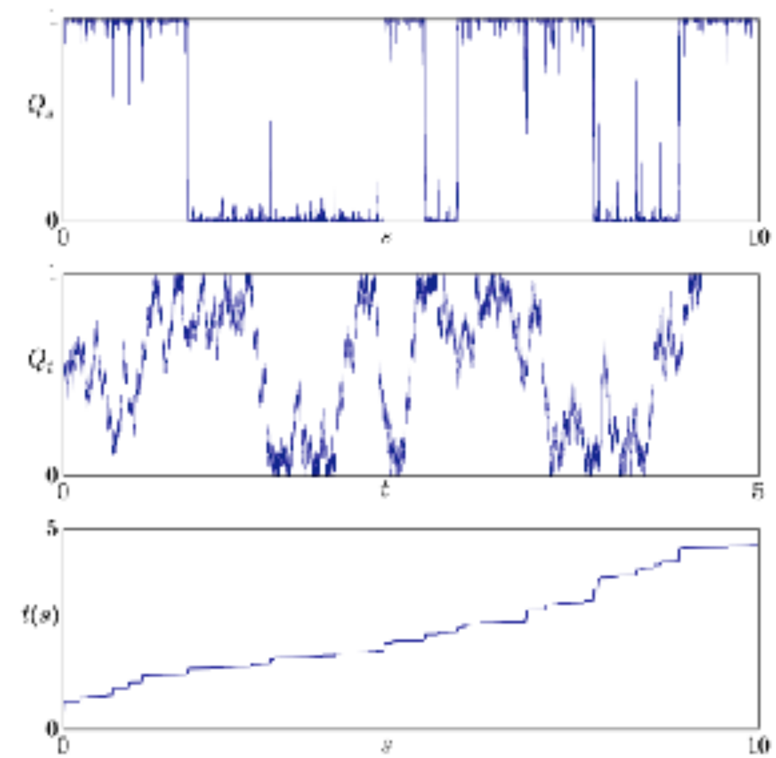


FIG. 2. *Continuous quantum trajectories in real and effective time. Top: The evolution of the ground state probability Q in real time s shows sharp jumps and spikes. Center: The evolution of Q in effective time t looks like a reflected Brownian motion without sharp transitions. Bottom: Effective time as a function of the real time. The plots are shown for the same realisation with* $\gamma = 200$ *(which looks like* $\gamma \to +\infty$*),* $p = 0.5$ *and* $\lambda = 1$.

$$\tau := \sum_n \mathrm{Tr}[(\delta\rho_n)^2] \quad \text{or} \quad \tau := \sum_n \mathrm{Tr}[(\delta\rho_n)^2_{\mathrm{diag}}]$$

— The question is how to reconstruct the physical time given the effective time and the signal…

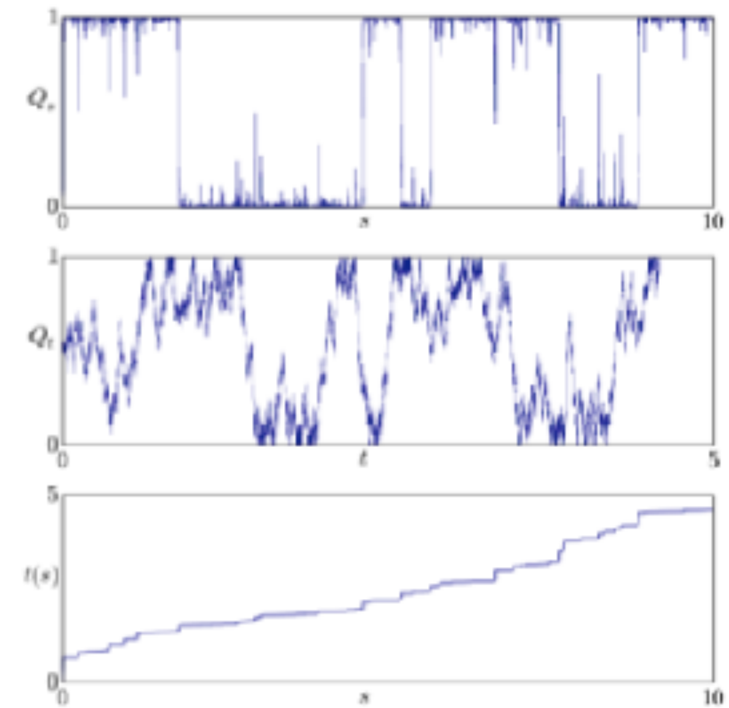
Internal structure of Q-jumps and Q-Spikes.

For a thermal Qu-bit, let $Q_t = \langle + | \rho_t | + \rangle$

— **Theorem**:

In the large monitoring rate limit, the Q-trajectories of thermal Qu-bit are equivalent `in law' to a Brownian motion reflected at 0 and 1 but parametrised by its local times via

$$Q_t = \text{const.}\ ||W_\tau|| \qquad \text{with} \qquad t = \frac{1}{\lambda p} L_\tau^{(0)} + \frac{1}{\lambda(1-p)} L_\tau^{(1)}$$



spiky process

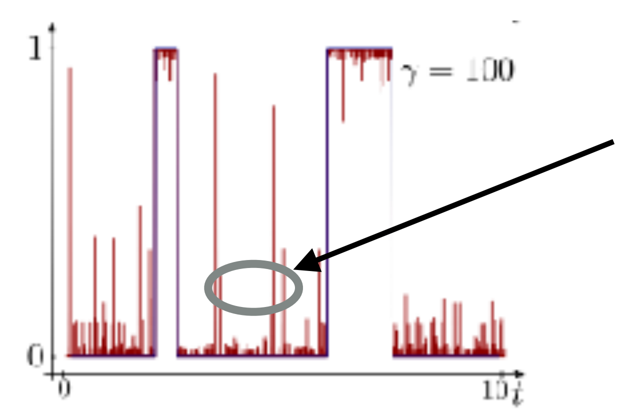
reflected Brownian

effective time

physical time

— **Proposition**:

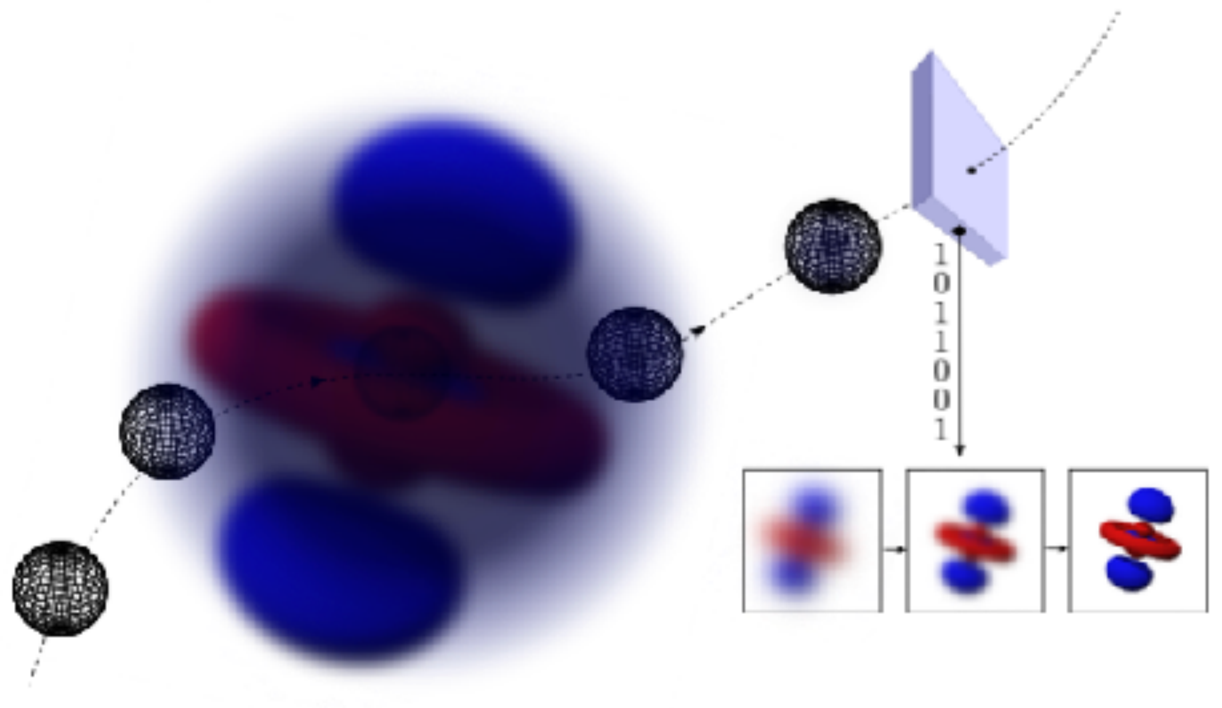
In the large monitoring rate limit,
- Quantum spikes have a scale invariant statistics;
- They form a Point Poisson Process



$$d\nu = \tilde{\lambda}\, \frac{dQ}{Q^2}\, dt$$

— Similar results hold for a (large) class of SDEs in the strong noise limit….

We stop here this series of lectures on
« Statistical Aspects of Quantum State Monitoring »
for (and by) Amateurs



THANK YOU!!

A finer structure: quantum spikes (II)

— **Why spikes survive?**

Look at the linear equation close to Q=0: $dX_t = \lambda p dt + \gamma X_t dB_t$.

Exact solution -> $\gamma^2 X_t$ converges in law:

$$\lambda p Y := \lim_{\gamma\to\infty} \gamma^2 X_t \qquad \text{with } \mathbb{P}[Y < y] = e^{-2/y}$$

In a given trajectory, all Xt are of order γ^{-2} but they are correlated on a time interval of order γ^{-2} . On a given time interval they are γ^2 such variables.

Hence, the maximum $M := \sup_{t\in[a,b]}[X_t]$ is of order 1:

$$\mathbb{P}[M < m] \simeq [e^{-2/\gamma^2 m}]^{\gamma^2(b-a)}$$

A finer structure: quantum spikes (II)

— **Check statistical property of Q-trajectories (derived from the SDEs)**

Poisson intensity for spikes emerging from 0: $d\nu_0 = \lambda p\, dt \cdot \left[\delta(1-Q)dQ + \frac{dQ}{Q^2}\right],$

— Recall: the distribution of the time a trajectory starting at Qi reaches Q is:

$$d\mathbb{P}_{\rm excur} := \frac{Q_i}{Q}\,\delta(t)dt + (1-\frac{Q_i}{Q})\frac{\lambda p}{Q}e^{-\frac{\lambda p}{Q}t}dt. \qquad \text{For } 0 < Q_i < Q < 1,$$

— This can be computed (« rederived »)using the spikes' description:

- Starting at Qi = conditioning on the presence of a spike point in [Qi,1]x[0,dt] (with dt->0)

  Two possibilities:

- This initial spike is actually above Q (first term)

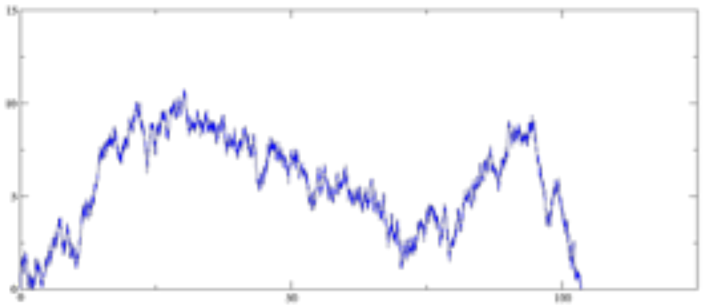
$$\mathbb{P}\left[\mathcal{N}_{[Q,1]\times[0,\delta t]} = 1 \middle| \mathcal{N}_{[Q_i,1]\times[0,\delta t]} = 1\right]\Big|_{\delta t\to 0}$$

- The next spikes above Q differ from the initial one (second term)

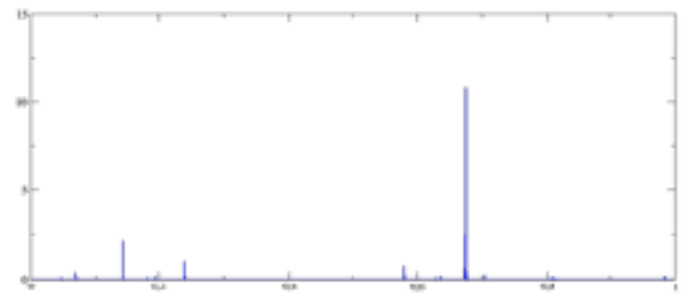
$$\mathbb{P}\left[\mathcal{N}_{[Q_i,Q]\times[0,\delta t]} = 1,\ \mathcal{N}_{[Q,1]\times[0,t]} = 0,\ \mathcal{N}_{[Q,1]\times[t,t+dt]} = 1 \middle| \mathcal{N}_{[Q_i,1]\times[0,\delta t]} = 1\right]\Big|_{\delta t\to 0}$$

An explicit construction at large coupling (II)

— Look again at the linear equation (close to Q=0): $dX_t = \lambda p dt + \gamma X_t dB_t.$
It develops spikes (from 0) at large gamma.



from natural parametrization → to local time parametrization



— **Hint for a proof:**

Change time (quadratic variation): new 'time' τ by $d\tau := \gamma^2 X_t^2 dt = (dX_t)^2$

Yield a new SDE($Z_\tau = X_t$) : $$dZ_\tau = \frac{\lambda p}{\gamma^2 Z_\tau^2} d\tau + dW_\tau = \lambda p\, dt_\tau + dW_\tau$$

Matters only at Z=0-> reflection

a (new) Brownian motion

Z has to be a Brownian motion reflected a zero —> $dZ_\tau = dL_\tau^{(0)} + dW_\tau$ [Tanaka formula]

Local time at 0

By identification: $dL_\tau^{(0)} = \lambda p\, dt$