QUANTUM MARKOV SYSTEMS

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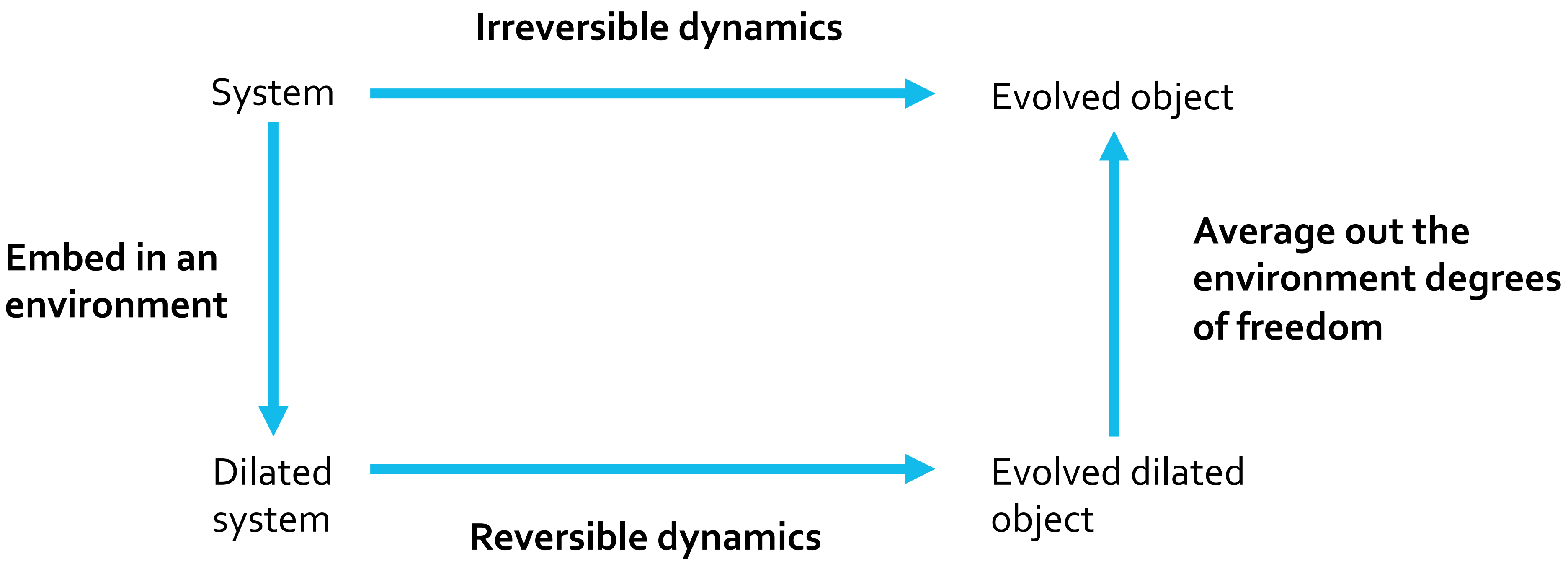
We have a simple scheme for obtained irreversible dynamics from reversible ones



3.1 Quantum System + Classical Noise

- Consider the stochastic unitary (*H*,*R* Hermitean) $U(t) = e^{-iHt-iRW(t)}.$

- We have the stochastic Schrödinger equation

$$dU(t) = \big[-iH - \frac{1}{2}R^2\big]U(t)\,dt - iRU(t)\,dW(t).$$

- Set $j_t(X) = U(t)^* X U(t),$ to get a stochastic Heisenberg equation

$$dj_t(X) = j_t\big(\mathcal{L}(X)\big)\,dt - ij_t\big([X,R]\big)\,dW(t).$$

where

$$\mathcal{L}(X) = -i[X,H] - \frac{1}{2}\big[[X,R],R\big].$$

Quantum System with Classical Jumps

- We can also apply a unitary kick *S* at random times: $$U(t) = S^{N(t)}$$

- Here *N(t)* is a Poisson process counting the number of kicks up to time *t*, it has independent increments and

$$dN(t)\,dN(t) = dN(t), \qquad \langle dN(t)\rangle = \nu\, dt.$$

- So

$$dj_t(X) = j_t\big(\mathcal{L}(X)\big)dN(t), \qquad \mathcal{L}(X) = S^*XS - X.$$

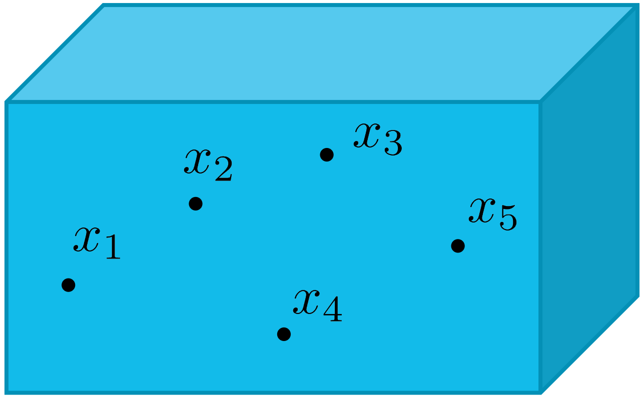
Lindblad Generators

- A quantum dynamical semigroup is a family of CP maps, $\{\Phi_t : t \geq 0\}$, such that $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi(I) = I$.
  The general form of the generator is

$$\mathcal{L}(X) = \sum_k \frac{1}{2} L_k^* [X, L_k] + \sum_k \frac{1}{2} [L_k^*, X] L_k - i[X, H].$$

- These include the examples emerging from classical noise, but the class of Lindblad generators is strictly larger that this – we need **quantum noise!**

3.1 Fock Space



- We consider a boson field: when we look at the field we see particles at specific locations.
- A pure state of the field … *n* particles

$$|\psi_n\rangle \quad \sim \psi_n(x_1, x_2, \cdots x_n).$$

The wave function is completely symmetric under interchange of labels.

- The number of particles in the state may be indefinite:

$$|\Psi\rangle = \big(\psi_0, \psi_1, \psi_2, \psi_3, \cdots\big).$$

Fock Space

- Note that $n = 0$ is included. No particles is a physical state of the field.
- The probability that we have exactly n particles is

$$p_n = \int \left|\psi_n\left(x_1, x_2, \cdots, x_n\right)\right|^2 dx_1 dx_2 \cdots dx_n.$$

Normalization is

$$\sum_{n=0}^{\infty} p_n = 1.$$

- The vacuum state is defined as

$$|\Omega\rangle = \big(1, 0, 0, 0, \cdots\big).$$

Fock Space

- The Hilbert space spanned by such indefinite number of indistinguishable boson states is called **Fock Space**.
- A total set is given by the **exponential vectors**

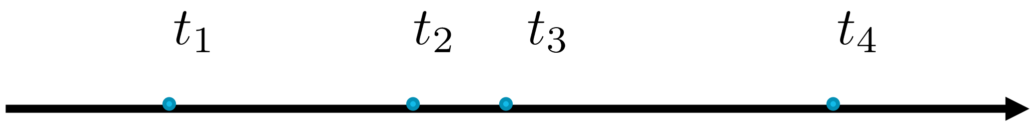
$$\langle x_1, x_2, \cdots, x_n | \exp(\alpha)\rangle = \frac{1}{\sqrt{n!}}\alpha(x_1)\,\alpha(x_2)\cdots\alpha(x_n)\,.$$

- They satisfy

$$\begin{aligned}\langle \exp(\alpha) | \exp(\beta)\rangle &= \sum_n \left(\frac{1}{\sqrt{n!}}\right)^2 \int \alpha(x_1)^* \cdots \alpha(x_n)^* \beta(x_1)\cdots\beta(x_n)\, dx_1 \cdots dx_n \\ &= e^{\int \alpha(x)^* \beta(x) dx} \\ &= e^{\langle\alpha|\beta\rangle}.\end{aligned}$$

- The vacuum is $|\Omega\rangle = |\exp(0)\rangle.$

3.3 Photons on a Wire



- Fock space over the segment [*s*,*t*] :- $\mathfrak{F}_{[s,t]}$.

$$\mathfrak{F}_{A\cup B} = \mathfrak{F}_A \otimes \mathfrak{F}_B, \qquad \text{if} A \cap B = \emptyset.$$

- Introduce quantum white noises

$$[b(t), b(s)^*] \quad = \quad \delta(t-s).$$

- *b(t)* annihilates a photon at *t*.

$$\begin{aligned} b(t)\,|\Omega\rangle &= 0 \\ b(t)\,|\exp(\beta)\rangle &= \beta(t)\,|\exp(\beta)\rangle. \end{aligned}$$

The Fundamental Quantum Stochastic Processes

- Creation/Annihilation $$B(t) = \int_0^t b(\tau)d\tau, \qquad B(t)^* = \int_0^t b(\tau)^* d\tau,$$

$$[B(t), B(s)^*] = \int_0^t d\tau \int_0^s d\sigma\, \delta(\tau - \sigma) = \min(t, s).$$

- Number (a.k.a. Gauge, conservation)

$$\Lambda(t) = \int_0^t b(\tau)^* b(\tau) d\tau.$$

Quantum Stochastic Processes

- Fix a system Hilbert space, $\mathfrak{h}_S$. A quantum stochastic process is a family of operators, $\{X(t): t\geq 0\}$, acting on $\mathfrak{h}_S \otimes \mathfrak{F}_{[0,\infty)}$.

- The process is adapted if, for each t, the operator *X(t)* acts trivially on the future environment factor $\mathfrak{F}_{[t,\infty)}$.

- QSDEs with adapted coefficients (Hudson & Parthasarathy, 1984)

$$
\begin{aligned}
\dot{X}(t) &= b(t)^*(t)X_{11}(t)b(t) + b(t)^*X_{10} + X_{01}(t)b(t) + X_{00}(t), \\
dX(t) &= X_{11}(t) \otimes d\Lambda(t) + X_{10}(t) \otimes dB(t)^* + X_{01}(t) \otimes dB(t) + X_{00}(t) \otimes dt,
\end{aligned}
$$

Quantum Stochastic Calculus

- $dB(t)dB(t) = dB(t)^* dB(t) = dB^*(t)dB^*(t) = 0$

  For X_t adapted

  $$\langle \exp(\alpha)| X_t dB(t)^* dB(t) | \exp(\beta)\rangle = \alpha(t)^* \langle \exp(\alpha)| X_t \exp(\beta)\rangle \beta(t)\,(dt)^2$$

- $dB(t)dB(t)^* = dt$

  $$[B(t) - B(s), B(t)^* - B(s)^*] = t - s, \qquad (t > s)$$

  so $\Delta B\,\Delta B^* = \Delta B^* \Delta B + \Delta t.$

Quantum Ito Table

- The full table is

$\times$	dt	dB	dB^*	$d\Lambda$
dt	0	0	0	0
dB	0	0	dt	dB
dB^*	0	0	0	0
$d\Lambda$	0	0	dB^*	$d\Lambda$

- Quantum Ito Rule

$$d\big(XY\big) = (dX)dY + dX(dY) + (dX)(dY).$$

Some "Classical" Processes

- The process $Q(t) = B(t) + B(t)^*$ is self-commuting, $[Q(t), Q(s)] = 0, \quad \forall t, s$ and had the distribution of a Wiener process is the vacuum state

$$\begin{aligned} \langle \dot{Q}(t) \rangle &= \langle \Omega | [b(t) + b(t)^*] \Omega \rangle = 0, \\ \langle \dot{Q}(t) \dot{Q}(s) \rangle &= \langle \Omega | b(t) b^*(s) \Omega \rangle = \delta(t - s). \end{aligned}$$

- The same applies to $P(t) = \frac{1}{i}[B(t) - B(t)^*]$, but ...

$$[Q(t), P(s)] = 2i \min(t, s).$$

- A Poisson process is given by

$$N(t) = \Lambda(t) + \sqrt{\nu} B^*(t) + \sqrt{\nu} B(t) + \nu t.$$

3.4 Emission - Absorption

- Hamiltonian

$$\Upsilon(t) = H \otimes I + iL \otimes b(t)^* - iL^* \otimes b(t).$$

- Unitary

$$\dot{U}(t) = -i\Upsilon(t)\,U(t), \qquad U(0) = I.$$

Not in Wick order!

- Dyson Series

$$\begin{aligned} U(t) &= I - i\int_0^t \Upsilon(\tau)U(\tau)d\tau \\ &= 1 - i\int_0^t d\tau\Upsilon(\tau) + (-i)^2\int_0^t d\tau_2\int_0^{\tau_2} d\tau_2\Upsilon(\tau_2)\Upsilon(\tau_1) + \cdots \\ &= \vec{T}e^{-i\int_0^t \Upsilon(\tau)d\tau} \end{aligned}$$

- Wick order the terms.

- One step re-ordering

$$
\begin{aligned}
[b(t), U(t)] &= \left[b(t), I - i \int_0^t \Upsilon(\tau) U(\tau) d\tau \right] = -i \int_0^t [b(t), \Upsilon(\tau)] U(\tau) d\tau \\
&= \int_0^t [b(t), Lb(\tau)^*] U(\tau) d\tau = L \int_0^t \delta(t-\tau) U(\tau) d\tau = \frac{1}{2} LU(t),
\end{aligned}
$$

so

$$
b(t) U(t) = U(t) b(t) + \frac{1}{2} LU(t).
$$

- The (Wick ordered) QSDE

$$
\begin{aligned}
\dot{U}(t) &= b(t)^* LU(t) - L^* b(t) U(t) - iH(t) U(t) \\
&= b(t)^* LU(t) - L^* U(t) b(t) - \left(\frac{1}{2} L^* L + iH \right) U(t).
\end{aligned}
$$

- The Hudson-Parthasarathy form is

$$dU(t) = \left\{ L \otimes dB(t)^* - L^* \otimes dB(t) - \left(\frac{1}{2} L^* L + iH \right) \otimes dt \right\} U(t) .$$

- The Heisenberg equation: $j_t(X) = U(t)^* [X \otimes I] U(t),$

$$\begin{aligned} dj_t(X) &= dU(t)^* [X \otimes I] U(t) + U(t)^* [X \otimes I] dU(t) + dU(t)^* [X \otimes I] dU(t) \\ &= j_t(\mathcal{L}X) \otimes dt + j_t([X, L]) \otimes dB(t)^* + j_t([L^*, X]) \otimes dB(t) \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}X &= -X \left(\frac{1}{2} L^* L + iH \right) - \left(\frac{1}{2} L^* L - iH \right) X + L^* X L \\ &= \frac{1}{2} [L^*, X] L + \frac{1}{2} L^* [X, L] - i [X, H] . \end{aligned}$$

.5 Scattering

Now try $\Upsilon(t) = E \otimes b(t)^* b(t)$.

$$[b(t), U(t)] = -iE \int_0^t [b(t), b(\tau)^*] b(\tau) U(\tau) d\tau = -\frac{i}{2} E b(t) U(t)$$

or

$$b(t) U(t) = \frac{1}{I - \frac{i}{2}E} U(t) b(t).$$

So

$$\dot{U}(t) = E b(t)^* b(t) U(t) = \frac{E}{I - \frac{i}{2}} b(t)^* U(t) b(t)$$

or in quantum Ito form

$$dU(t) = (S - I) \otimes d\Lambda(t)\, U(t), \qquad \left(S = \frac{I + \frac{i}{2}E}{I - \frac{i}{2}E}, \text{ unitary!} \right).$$

The Heisenberg equation here is $dj_t(X) = j_t(S^* X S - X) \otimes d\Lambda(t)$.

3.6 *SLH* Formalism

- Quantum white noises

$$[b_j(t), b_k^*(s)] = \delta_{jk}\,\delta(t-s)$$

- Hamiltonian *H*

$$H^* = H$$

- Coupling/Collapse Operators *L*

$$L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

- Scattering Operator *S*

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \qquad S^{-1} = S^*$$

Quantum Stochastic Models

- General (S, L, H) case

Wick-ordered form:

$$dU(t) = \Big\{ \sum_{jk} (S_{jk} - \delta_{jk} I) \otimes d\Lambda_{jk}(t) + \sum_j L_j \otimes dB_j^*(t) - \sum_{jk} L_j^* S_{jk} \otimes dB_k(t) - \big(\frac{1}{2} \sum_k L_k^* L_k + iH\big) \otimes dt \Big\} U(t)$$

Heisenberg Picture

$$dj_t(X) = \sum_{jk} j_t(S_{lj}^* X S_{lk} - \delta_{jk} X) d\Lambda_{jk}(t) + \sum_{jl} j_t(S_{lj}^*[L_l, X]) \otimes dB_j(t) + \sum_{lk} j_t([X, L_l^*] S_{lk}) \otimes dB_k(t) + j_t(\mathscr{L} X) \otimes dt.$$

Lindblad Generator

$$\mathscr{L} X = \tfrac{1}{2} \sum_k L_k^* [X, L_k] + \tfrac{1}{2} \sum_k [L_k^*, X] L_k - i[X, H]$$

Input-Output Relations

- The outputs are defined by

$$B_k^{\mathrm{out}}(t) = U(t)^*[I \otimes B_k(t)]U(t).$$

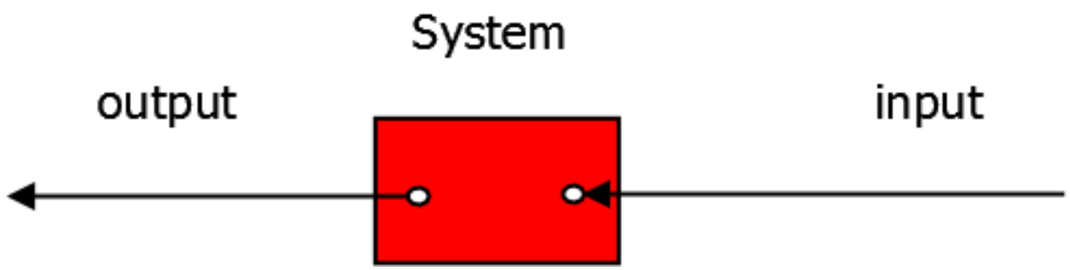
- From the quantum Ito calculus

$$dB_j^{\mathrm{out}}(t) = \sum_k j_t(S_{jk}) \otimes dB_k(t) + j_t(L_k) \otimes dt,$$

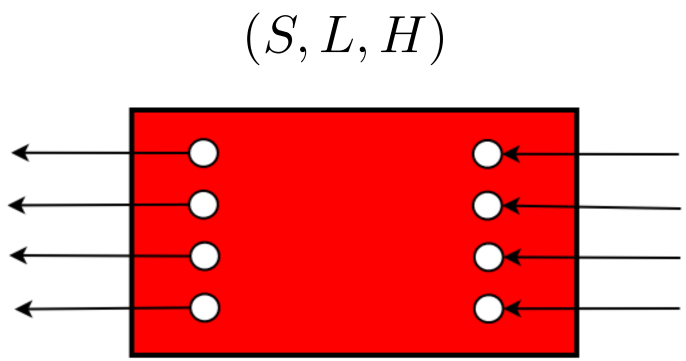
Or,

$$b_j^{\mathrm{out}}(t) = \sum_j j_t(S_{jk}) \otimes b_k(t) + j_t(L_j) \otimes I.$$

Quantum Markovian Models



The "wires" are quantum fields and may carry a multiplicity.



3.6 Quantum Filtering

Let us measure the quadrature $Q(t) = B(t) + B(t)^*$. Set

$$Y^{\text{in}}(t) = I \otimes Q(t).$$

The initial state is taken to be $|\phi\rangle \otimes |\Omega\rangle$. In the Heisenberg picture, this state is fixed and the observables evolve

$$\begin{aligned} j_t(X) &= U(t)^* [X \otimes I] U(t), \\ Y^{\text{out}}(t) &= U(t)^* [I \otimes Q(t)] U(t). \end{aligned}$$

The unitary

$$U(t,s) = Te^{-i\int_s^t \Upsilon(\tau)d\tau}$$

couples the system to the part of the field that enters over the time $s \leq \tau \leq t$. We have

$$U(t) = U(t,s)\,U(s)\,, \qquad (t > s > 0)\,.$$

A key identity is that

$$Y^{\text{out}}(s) = U(t)^* \, Y^{\text{in}}(s)\, U(t)\,, \qquad (t > s)\,.$$

This follows from the fact that $\left[Y^{\text{in}}(s)\,, U(t,s)\right] = 0$.

From this, we see that the process Y^{out} is also commutative

$$\left[Y^{\text{out}}(t), Y^{\text{out}}(s)\right] = U(t)^* \left[Y^{\text{in}}(t), Y^{\text{in}}(s)\right] U(t) = 0, \quad (t > s).$$

Also

$$\left[j_t(X), Y^{\text{out}}(s)\right] = U(t)^* \left[X \otimes I, I \otimes Q(t)\right] U(t) = 0, \quad (t > s).$$

We can have a joint probability for $j_t(X)$ and the $\{Y^{\text{out}}(\tau) : 0 \leq \tau \leq t\}$ so can use Bayes Theorem.

Note that

$$\begin{aligned} dY^{\text{out}}(t) &= dB^{\text{out}}(t) + dB^{\text{out}}(t)^* \\ &= dY^{\text{in}}(t) + j_t(L + L^*)\, dt \end{aligned}$$

and

$$dY^{\text{in}}(t)\, dY^{\text{in}}(t) = dt = dY^{\text{out}}(t)\, dY^{\text{out}}(t).$$

The state at time t is $|\Psi_t\rangle = U(t)|\phi \otimes \Omega\rangle$, so

$$\begin{aligned} d|\Psi_t\rangle &= -\left(\frac{1}{2}L^*L + iH\right)|\Psi_t\rangle dt + LdB(t)^*|\Psi_t\rangle - L^*dB(t)|\Psi_t\rangle \\ &= -\left(\frac{1}{2}L^*L + iH\right)|\Psi_t\rangle dt + LdB(t)^*|\Psi_t\rangle + LdB(t)|\Psi_t\rangle \\ &= -\left(\frac{1}{2}L^*L + iH\right)|\Psi_t\rangle dt + LdQ_t|\Psi_t\rangle. \end{aligned}$$

Which is equivalent to the SDE in the system Hilbert space

$$d|\chi_t\rangle = -\left(\frac{1}{2}L^*L + iH\right)|\chi_t\rangle dt + L|\chi_t\rangle dy_t$$

where $\mathbf{y}$ is a sample path.

We have

$$
\begin{aligned}
\langle \phi \otimes \Omega | j_t\left(X\right) F\left[Y^{\mathrm{out}}_{[0,t]} \right] | \phi \otimes \Omega \rangle &= \langle \phi \otimes \Omega | U(t)^* \left(X \otimes F\left[Y^{\mathrm{in}}_{[0,t]} \right] \right) U(t) | \phi \otimes \Omega \rangle \\
&= \langle \Psi_t | X \otimes F\left[Y^{\mathrm{in}}_{[0,t]} \right] | \Psi_t \rangle \\
&= \int \langle \chi_t(\mathbf{y}) | X \otimes | \chi_t(\mathbf{y}) \rangle \, F\left[\mathbf{y} \right] \, \mathbb{P}_{\mathrm{Wiener}}[d\mathbf{y}]
\end{aligned}
$$

Setting $X = I$, we get the

$$\langle \phi \otimes \Omega | F \left[Y^{\text{out}}_{[0,t]} \right] | \phi \otimes \Omega \rangle \quad = \quad \int \langle \chi_t(\mathbf{y}) | \chi_t(\mathbf{y}) \rangle \, F\left[\mathbf{y}\right] \, \mathbb{P}_{\text{Wiener}}[d\mathbf{y}]$$

So the probability of the measured paths is

$$\mathbb{Q}[d\mathbf{y}] = \langle \chi_t(\mathbf{y}) | \chi_t(\mathbf{y}) \rangle \, \mathbb{P}_{\text{Wiener}}[d\mathbf{y}]$$

And we also determine the filter (using the arbitrariness of F)

$$\pi_t(X) = \frac{\langle \chi_t(\mathbf{y}) | X | \chi_t(\mathbf{y}) \rangle}{\langle \chi_t(\mathbf{y}) | \chi_t(\mathbf{y}) \rangle}.$$

We see that (using $dy(t)\, dy(t) = dt$)

$$d\langle \chi_t(\mathbf{y}) | X | \chi_t(\mathbf{y}) \rangle = \langle \chi_t(\mathbf{y}) | \mathcal{L} X | \chi_t(\mathbf{y}) \rangle dt + \langle \chi_t(\mathbf{y}) | X L + L^* X | \chi_t(\mathbf{y}) \rangle\, dy(t).$$

Same form as derived by Denis and Peter!