Computational aspects of robust optimized certainty equivalent

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Motivation

Let $X : \Omega \to \mathbb{R}$ be a future **uncertain** loss with distribution μ_X; and $l : \mathbb{R} \to \mathbb{R}$ a loss function.

$$\mathrm{OCE}(X) = \inf_{m \in \mathbb{R}} \left(\int l(x - m)\, d\mu_X + m \right) \equiv \mathrm{OCE}(\mu_X)$$

Ben-Tal & Teboulle (1986, 2007);

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Ben-Tal & Teboulle (1986, 2007); Cheridito & Li (2009); Cherny & Kupper (2007); Barrieu & El Karoui (2007); Drapeau et. al (2014); Backhoff & T. (2016)

- Expected loss of X (w.r.t. loss function l):

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$$\int l(x+m)\, d\mu_X - m$$

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- Minimize the allocation cost:

$$OCE(\mu_X) := \inf_{m\in\mathbb{R}} \left(\int l(x-m)\,d\mu_X + m \right)$$

- $l(x) = e^x - 1 \quad \rightsquigarrow \quad$ Entropy:

$$\mathrm{OCE}(\mu_X) = \mathrm{Ent}(\mu_X) := \log \int e^x \, d\mu_X$$

- $l(x) = x^+/\alpha \quad \rightsquigarrow \quad$ Average value-at-risk:

$$\mathrm{OCE}(\mu_X) = \mathrm{AVaR}_\alpha(\mu_X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_u(\mu_X) du$$

- In practice μ_X is not precisely known
- If the distribution $\mu_X \in \mathcal{D}$ for some $\mathcal{D} \subseteq \mathcal{M}_1$, then

$$\mathrm{OCE}(\mathcal{D}) := \inf_m \sup_{\mu \in \mathcal{D}} \left(\int l(x-m)\, d\mu + m \right) \equiv \mathrm{OCE}(\mathcal{D}, l)$$

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- Monetary risk measure? robust representation? computational issues? deviation from non-robust counterpart?

Optimal transport-type ambiguity sets

Let $c : \mathbb{R} \times \mathbb{R} \to [0, \infty]$ be lsc. For $\delta > 0$ and a baseline distribution $\mu_0 \in \mathcal{M}_1$, put

$$B_\delta(\mu_0) := \{\mu \in \mathcal{M}_1 : d_c(\mu_0, \mu) \leq \delta\}$$

with the "transport-like distance"

$$d_c(\mu, \nu) := \inf \left\{ \int c \, d\pi : \pi \in \mathcal{M}_1(\mathbb{R}^2) \text{ with } \pi(\cdot \times \mathbb{R}) = \mu, \; \pi(\mathbb{R} \times \cdot) = \nu \right\}.$$

Consider:

$$\mathrm{OCE}(B_\delta(\mu_0)) = \inf_m \sup_{\mu \in B_\delta(\mu_0)} \left(\int l(x-m)\, d\mu + m \right)$$

- If $c(x,y) = 1_{x \neq y}$, $2d_c = TV$, the total variation
- If $c(x,y) = |x-y|^p$, $d_c^{1/p} = W_p$, the p-Wassestein distance.
- d_c is a "better distance" than ϕ-divergences as it does not require absolute continuity.

  cf. e.g. Glasserman & Xu (2014); Hansen & Sargen (2001); Jian & Guan (2015).

Theorem

*If $l : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is **measurable and bounded from below**, assume $c(x, y) = \tilde{c}(x - y)$ for some $\tilde{c} : \mathbb{R} \to [0, \infty]$ s.t.*

- $\inf_x \tilde{c}(x) = 0$
- $\liminf_{|x| \to +\infty} \tilde{c}(x) = +\infty$.

Then,

$$OCE(B_\delta(\mu_0), l) = \inf_{\lambda \geq 0} \left(OCE(\mu_0, l^{\lambda c}) + \lambda \delta \right),$$

with $l^c(x) := \sup_{y \in \mathbb{R}} (l(y) - c(x, y))$, the c-transform of l.

Example

- Average value-at-risk For $\ell(x) = x^+/\alpha$ and $c(x, y) = |y - x|$; $d_c = 1-$Wasserstein distance

$$\mathrm{AVaR}_\alpha(B_\delta(\mu_0)) = \mathrm{AVaR}_\alpha(\mu_0) + \frac{\delta}{\alpha}$$

Thank You!