Super-replication with proportional transaction cost under model uncertainty

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Outline

No-arbitrage and FTAP

• Let us consider a one period market on a finite probability space $\Omega = \{\omega_1, \cdots, \omega_n\}$, s.t. $\mathbb{P}(\{\omega_i\}) > 0$ for each i. One has a tradable risky asset $S = (S_0, S_1)$, where $S_0 = s_0 \in \mathbb{R}_+$ is a constant, $S_1 : \Omega \to \mathbb{R}_+$. The interest rate $r = 0$.

• No arbitrage condition (NA) :

$$H(S_1 - S_0) \geq 0, \text{ a.s. } \Rightarrow H(S_1 - S_0) = 0, \text{ a.s.}$$

• Fundamental Theorem of Asset Pricing (FTAP) : (i) (NA) is equivalent to the existence of equivalent martingale measure, i.e.

$$\mathcal{M} := \{\mathbb{Q} \sim \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[S_1] = S_0\} \neq \emptyset.$$

(ii) The market is complete iff $\mathcal{M}$ is a singleton.

• Option pricing by replication if the market is complete..

Super-replication and the duality

• Incomplete market : super-replication problem for a derivative option $\xi : \Omega \to \mathbb{R}$:

$$\pi(\xi) \quad := \quad \inf \left\{ y \in \mathbb{R} \ : y + H(S_1 - S_0) \geq \xi, \ \text{a.s.} \right\}$$

• The superhedging duality :

$$\pi(\xi) \ = \ \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi].$$

• Superhedging duality, optional decomposition, martingale optimal transport : Kantorovich duality, etc.

Super-replication under proportional transaction cost

- Super-replication without transaction cost :

$$\begin{aligned}\pi(\xi) &:= \inf\left\{y\in\mathbb{R}\ : y + H(S_1 - S_0) \geq \xi,\ \text{a.s.}\right\}\\ &= \inf\left\{y\in\mathbb{R}\ : y \geq \xi - H(S_1 - S_0),\ \text{a.s.}\right\}\\ &= \inf_{H\in\mathbb{R}}\ \sup_{\omega}\ \left\{\xi(\omega) - H(S_1(\omega) - S_0)\right\}.\end{aligned}$$

- Super-replication under proportional transaction cost : Assume a proportional cost at time 1, then

$$\begin{aligned}\pi_c(\xi) &:= \inf\left\{y\in\mathbb{R}\ : y + H(S_1 - S_0) - |H|cS_1 \geq \xi,\ \text{a.s.}\right\}\\ &= \inf_{H\in\mathbb{R}}\ \sup_{\omega}\ \left\{\xi(\omega) - H(S_1(\omega) - S_0) + |H|cS_1(\omega)\right\}.\end{aligned}$$

Super-replication under proportional transaction cost

- Reformulation : Notice that $|H| = \sup_{\theta\in\{-1,1\}} H\theta$, then

$$\begin{aligned}
\pi_c(\xi) &= \inf_{H\in\mathbb{R}} \sup_{\omega\in\Omega} \left\{\xi(\omega) - H(S_1(\omega) - S_0) + |H|cS_1(\omega)\right\} \\
&= \inf_{H\in\mathbb{R}} \sup_{\omega\in\Omega} \sup_{\theta\in\{-1,1\}} \left\{\xi(\omega) - HS_1(\omega) + H\theta cS_1(\omega) + HS_0\right\} \\
&= \inf_{H\in\mathbb{R}} \sup_{\bar\omega\in\overline{\Omega}} \left\{\xi(\omega) - HX_1(\bar\omega) + HS_0\right\},
\end{aligned}$$

where $\overline{\Omega} := \{\omega_1, \cdots, \omega_n\} \times \{-1, 1\}$, and

$$X_1(\bar\omega) := X_1(\omega, \theta) = (1 - \theta c)S_1(\omega), \quad X_0 := S_0.$$

Then

$$\pi_c(\xi) = \inf\left\{y \in \mathbb{R} : y + H(X_1 - X_0) \geq \xi, \text{ a.s.}\right\}.$$

Super-replication under proportional transaction cost

• The duality : (under the (NA) condition)

$$\pi_c(\xi) := \inf\left\{y \in \mathbb{R} \ : y + H(S_1 - S_0) - |H|cS_1 \geq \xi, \ \text{a.s.}\right\}$$
$$= \inf\left\{y \in \mathbb{R} \ : y + H(X_1 - X_0) \geq \xi, \ \text{a.s.}\right\}$$
$$= \sup\left\{\mathbb{E}^{\overline{\mathbb{Q}}}[\xi] \ : \ \overline{\mathbb{Q}} \sim \overline{\mathbb{P}} \text{ s.t. } \mathbb{E}^{\overline{\mathbb{Q}}}[X_1] = X_0 := S_0\right\}.$$

• Remark : By construction, one has $X_1 \in \big[(1-c)S_1, (1+c)S_1\big]$. Then $(X, \mathbb{Q})$ is a consistent price system when (X_0, X_1) is a $\mathbb{Q}$-martingale.

Outline

Super-replication under proportional transaction cost

• Bouchard-Nutz's framework : $\Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_1$, a family of probability measures $\mathcal{P} := \{\mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t \in \mathcal{P}_t(\cdot)\}$.

• Transaction cost modelled by the solvency cone $(K_t(\omega))_{0 \le t \le T}$.

• Possible vanilla options for semi-statique strategies.

• Main result : Under the a robust no-arbitrage condition introduced in Bouchard-Nutz (2016), one has the duality :

$$\begin{aligned} & \text{minimum super-hedging cost of } \xi \\ = \ & \sup\left\{\mathbb{E}^{\mathbb{Q}}[\xi] \ : \text{all consistent price system } (\mathbb{Q}, X)\right\}. \end{aligned}$$

References

- Dolinsky and Soner (2014) : Duality in the MOT context with small transaction cost, $d = 1$.
- Bayraktar and Zhang (2016) : FTAP in a quasi-sure context.
- Bouchard and Nutz (2016) : FTAP in a quasi-sure context.
- Burzoni (2016) : FTAP and duality in a pointwise hedging context, simple transaction cost model.
- Bouchard, Deng and Tan (2017) : Duality result in a quasi-sure context.