Sensitivity analysis of the expected utility maximization problem with respect to model perturbations

Mihai Sîrbu, The University of Texas at Austin

based on joint work with

Oleksii Mostovyi
University of Connecticut

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Outline

The starting point

- consider the perturbation analysis in Larsen, Mostovyi and Žitković
- do a similar analysis with a general rather than power utility

The mathematics

- present a method to approximate
  1. value functions to second order
  2. optimizers to the first order
- stochastic control problems which are convex, but not convex with respect to a parameter
- abstract version (over random variables)
- back to the original model, write approximation of strategies as Kunita-Watanabe decomposition under risk tolerance wealth process as numeraire

Stability and asymptotics

Existing results (small fraction):

- dependence on x: Kramkov and Schachermayer (1999, 2003), Kramkov and S. (2006),
- dependence on U (and/or $\mathbb{P}$): Jouini and Napp (2004), Carasus and Rasonyi (2005), Larsen (2006), Kardaras and Žitković (2011),
- dependence on the parametrization of the stock price: Prigent (2003), Larsen and Žitković (2007), Larsen, Mostovyi, and Žitković (2014).
- on random endowment: Henderson (2002), Kramkov and Sîrbu (2006, 2007), Kallsen, Muhle-Karbe, and Vierthauer (2014).

The family of markets

(from Larsen, Mostovyi, Žitković)

A family of markets is parametrized by δ.
Every market consist of a stock and a bond

$$S_t^{\delta} \triangleq M_t + \int_0^t (\lambda_s + \delta\nu_s)\, d\langle M\rangle_s, \quad t \in [0, T],$$

(see Hulley and Schweizer (2010), Delbaen and Schachermayer (1995));

The price process of the bond equals to 1 at all times.

Primal problem

Define

$$\mathcal{X}(x,\delta) \triangleq \left\{ X : \; X_t = x + \int_0^t H_u dS_u^{\delta}, t \in [0,T] \text{ and } X \geq 0 \right\},$$
$$x > 0.$$

A utility function $U : (0,\infty) \to \mathbb{R}$ is strictly increasing, strictly concave, two times continuously differentiable on $(0,\infty)$ and there exist positive constants c_1 and c_2, such that

$$c_1 \leq A(x) \triangleq -\frac{U''(x)x}{U'(x)} \leq c_2,$$

and define the value function as:

$$u(x,\delta) \triangleq \sup_{X \in \mathcal{X}(x,\delta)} \mathbb{E}\left[U(X_T)\right], \quad (x,\delta) \in (0,\infty) \times \mathbb{R}.$$

Mathematical goal

How to establish an expansion with respect to δ of

- the value function $u(x, \delta)$,
- the corresponding trading strategy?

Remark

Dual problem can be helpful.

Dual problem

$$V(y) \triangleq \sup_{x>0} \left(U(x) - xy \right), \quad y > 0,$$
$$-\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}, \quad \textit{if} \quad y = U'(x).$$

Let $\mathcal{Y}(y, \delta)$ be a set of nonnegative supermartingales such that:

1. $Y_0 = y$,
2. $(X_t Y_t)_{t \in [0,T]}$ is a supermartingale for every $X \in \mathcal{X}(1, \delta)$.

The dual value function is

$$v(y, \delta) \triangleq \inf_{Y \in \mathcal{Y}(y,\delta)} \mathbb{E}\left[V(Y_T) \right], \quad (y, \delta) \in (0, \infty) \times \mathbb{R}.$$

Structural lemma

Lemma

For every $\delta \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{Y}(1,\delta) &= \mathcal{Y}(1,0)\mathcal{E}\left(-\delta\nu \cdot S^0\right),\\ \mathcal{X}(1,\delta) &= \mathcal{X}(1,0)\frac{1}{\mathcal{E}\left(-\delta\nu \cdot S^0\right)}.\end{aligned}$$

Remark

Looks like a multiplicative (and non-linear) random endowment.

Abstract theorems

In the spirit of Kramkov-Schachermayer (99), consider the sets $\mathcal{C}$ and $\mathcal{D}$ polar in $\mathbf{L}^0_+$:

Assumption

Both $\mathcal{C}$ and $\mathcal{D}$ contain a stricly positive element and

$$\xi \in \mathcal{C} \quad \textit{iff} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \textit{for every } \eta \in \mathcal{D},$$

as well as

$$\eta \in \mathcal{D} \quad \textit{iff} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \textit{for every } \xi \in \mathcal{C}.$$

Primal and dual problems for 0-model

We set

$$\mathcal{C}(x,0) \triangleq x\mathcal{C} \quad \textit{and} \quad \mathcal{D}(x,0) \triangleq x\mathcal{D}, \quad x > 0.$$

Now we can state the abstract primal and dual problems as

$$u(x,0) \triangleq \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\left[U(\xi)\right], \quad x > 0,$$

$$v(y,0) \triangleq \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}\left[V(\eta)\right], \quad y > 0.$$

Abstract version for δ-models

For some random variables F and $G \geq 0$, we set

$$L^{\delta} \triangleq \exp\left(-(\delta F + \tfrac{1}{2}\delta^{2} G)\right),$$

$$\mathcal{C}(x,\delta) \triangleq \mathcal{C}(x,0)\frac{1}{L^{\delta}} \quad \textit{and} \quad \mathcal{D}(y,\delta) \triangleq \mathcal{D}(y,0)L^{\delta}, \quad \delta \in \mathbb{R}.$$

The abstract versions of the perturbed optimization problems:

$$u(x,\delta) \triangleq \sup_{\xi \in \mathcal{C}(x,\delta)} \mathbb{E}\left[U(\xi)\right] = \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\left[U\left(\xi \frac{1}{L^{\delta}}\right)\right], \quad x > 0, \delta \in \mathbb{R},$$

$$v(y,\delta) \triangleq \inf_{\eta \in \mathcal{D}(y,\delta)} \mathbb{E}\left[V(\eta)\right] = \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}\left[V\left(\eta L^{\delta}\right)\right], \quad y > 0, \delta \in \mathbb{R}.$$

The approach

Follows Henderson (2002).

- find lower bound up to second order for u
- upper bound up to second order for v
- "match" them

Matching one-sided bounds can be found using quadratic optimization (Kramkov, S. (2006))

Issue: no-convexity in δ. Can use convexity only in direction of x. For

$$y = u_x(x, \delta), \quad u(x, \delta) - xy = v(y, \delta)$$

Even if we fix x and vary δ alone, y depends on δ: need to approximate at least v in both directions (y, δ).
Summary: better provide joint expansion for both

- (x, δ) for u
- (y, δ) for v

The 0-model:

If u is finite at some point

(i) $u(x,0) < \infty$, for every $x > 0$, and $v(y,0) > -\infty$, for every $y > 0$. The functions u and v are Legendre conjugate

$$\begin{aligned} v(y,0) &= \sup_{x>0} \left(u(x,0) - xy \right), \quad y > 0, \\ u(x,0) &= \inf_{y>0} \left(v(y,0) + xy \right), \quad x > 0. \end{aligned}$$

(ii) The functions u and $-v$ are continuously differentiable on $(0,\infty)$, strictly concave, strictly increasing and satisfy the Inada conditions

$$\begin{aligned} \lim_{x\downarrow 0} u_x(x,0) &= \infty, & \lim_{y\downarrow 0}(-v_x(y,0)) &= \infty, \\ \lim_{x\uparrow \infty} u_x(x,0) &= 0, & \lim_{y\uparrow \infty}(-v_y(y,0)) &= 0. \end{aligned}$$

(iii) For every $x > 0$ and $y > 0$, the solutions $\widehat{X}(x,0)$ and $\widehat{Y}(y,0)$ exist and are unique and, if $y = u'(x)$, we have

$$\widehat{Y}_T(y) = U'\left(\widehat{X}_T(x)\right), \quad \mathbb{P}\text{-a.s.}$$

Assumption on perturbations

- First, we set:

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} \triangleq \frac{\widehat{X}_T(x,0)\widehat{Y}_T(y,0)}{xy}.$$

- Let $x > 0$ be fixed. There exists $c > 0$, such that

$$\mathbb{E}^{\mathbb{R}(x,0)}\left[\exp\left(c(|\nu \cdot S^0_T| + \langle \nu \cdot S^0 \rangle_T)\right)\right] < \infty.$$

Assumption under $\mathbb{P}$ and original numéraire

- Let us assume that $c_1 > 1$, i.e. that relative-risk aversion of U is strictly greater than 1 (relative risk aversion uniformly exceeds 1)
- A sufficient condition for the previous slide Assumption to hold is the existence of some positive exponential moments under $\mathbb{P}$

First-order analysis

Theorem (Envelope)

Let $x > 0$ be fixed and assumptions above hold. Then we have

- *There exists $\delta_0 > 0$ such that for every $\delta \in (-\delta_0, \delta_0)$, we have*
$$u(z, \delta) \in \mathbb{R} \quad \text{and} \quad v(z, \delta) \in \mathbb{R}, \quad z > 0.$$

- *The first-order derivatives are*
$$u_\delta(x, 0) = v_\delta(y, 0) = xy\mathbb{E}^{\mathbb{R}(x,0)}\left[\nu \cdot S_T^0\right], \quad y = u_x(x, 0).$$

- *The value functions u and v are continuous at $(x, 0)$ and $(y, 0)$, respectively.*

Remark

$u_\delta(x, 0)$ and $v_\delta(y, 0)$ are linear in ν.

Second-order analysis

- Let $S^{X(x,0)}$ be the price process of the traded securities under the numéraire $\frac{\widehat{X}(x,0)}{x}$, i.e.

$$S^{X(x,0)} = \left(\frac{x}{\widehat{X}(x,0)}, \frac{xS^0}{\widehat{X}(x,0)} \right).$$

- For every $x > 0$, let $\mathbf{H}_0^2(\mathbb{R}(x,0))$ denote the space of square integrable martingales under $\mathbb{R}(x,0)$, such that

$$\begin{aligned} \mathcal{M}^2(x,0) &\triangleq \left\{ M \in \mathbf{H}_0^2(\mathbb{R}(x,0)) : M = H \cdot S^{X(x,0)} \right\}, \\ \mathcal{N}^2(y,0) &\triangleq \left\{ N \in \mathbf{H}_0^2(\mathbb{R}(x,0)) : MN \textit{ is } \mathbb{R}(x,0)\text{-martingale} \right. \\ &\qquad \left. \text{for every } M \in \mathcal{M}^2(x,0) \right\}, \quad y = u_x(x,0). \end{aligned}$$

Auxiliary minimization problems (for u_{xx} and v_{yy})

Let us set

$$a(x,x) \triangleq \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[A(\widehat{X}_T(x,0))(1+M_T)^2 \right],$$

$$b(y,y) \triangleq \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[B(\widehat{Y}_T(y,0))(1+N_T)^2 \right],$$

where $y = u_x(x,0)$,

$$A(x) = -\frac{U''(x)x}{U'(x)} \quad \textit{and} \quad B(y) = -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}.$$

Second-order derivatives with respect to x and y

Proved in Kramkov and S. (2006):

- auxiliary minimization problems admit unique solutions $M^0(x,0)$ and $N^0(y,0)$;
- the value functions are two-times differentiable and

$$\begin{aligned} u_{xx}(x,0) &= -\tfrac{y}{x}a(x,x), \\ v_{yy}(y,0) &= \tfrac{x}{y}b(y,y); \end{aligned}$$

- u_{xx} and v_{yy} are linked via

$$\begin{aligned} u_{xx}(x,0)v_{yy}(y,0) &= -1, \\ a(x,x)b(y,y) &= 1; \end{aligned}$$

- the optimizers to auxiliary problems satisfy

$$A(\widehat{X}_T(x,0))(1+M^0_T(x,0)) = a(x,x)(1+N^0_T(y,0)).$$

Auxiliary minimization problem (for $u_{\delta\delta}$ and $v_{\delta\delta}$)

With

$$F \triangleq \nu \cdot S_T^0 \quad \textit{and} \quad G \triangleq \nu^2 \cdot \langle M \rangle_T,$$

we consider the following minimization problems.

$$a(d,d) \triangleq \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \Big[A(\widehat{X}_T(x,0))(M_T + xF)^2 - 2xFM_T - x^2(F^2 + G) \Big],$$

$$b(d,d) \triangleq \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \Big[B(\widehat{Y}_T(y,0))(N_T - yF)^2 + 2yFN_T - y^2(F^2 - G) \Big].$$

Structure of $u_{x\delta}$ and $v_{y\delta}$

Denoting by $M^1(x,0)$ and $N^1(y,0)$ the unique solutions the auxiliary problems above, we set

$$a(x,d) \triangleq \mathbb{E}^{\mathbb{R}(x,0)}\Big[A(\widehat{X}_T(x,0))(1+M_T^0(x,0))(xF+M_T^1(x,0)) \\ -xF(1+M_T^0(x,0))\Big],$$

$$b(y,d) \triangleq \mathbb{E}^{\mathbb{R}(x,0)}\Big[B(\widehat{Y}_T(y,0))(1+N_T^0(y,0))(N_T^1(y,0)-yF) \\ +yF(1+N_T^0(y,0))\Big].$$

Theorem (Mostovyi., S.)

Let $x > 0$ be fixed. Let the assumptions above hold and $y = u_x(x,0)$. Define

$$H_u(x,0) \triangleq -\frac{y}{x}\begin{pmatrix} a(x,x) & a(x,d) \\ a(x,d) & a(d,d) \end{pmatrix},$$

$$H_v(y,0) \triangleq \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,d) \\ b(y,d) & b(d,d) \end{pmatrix}.$$

Then, the value functions u and v admit the second-order expansions around $(x,0)$ and $(y,0)$, respectively,

$$\begin{aligned} u(x+\Delta x,\delta) &= u(x,0) + (\Delta x \quad \delta)\nabla u(x,0) \\ &\quad + \tfrac{1}{2}(\Delta x \quad \delta) H_u(x,0)\begin{pmatrix}\Delta x \\ \delta\end{pmatrix} + o(\Delta x^2 + \delta^2), \\ v(y+\Delta y,\delta) &= v(y,0) + (\Delta y \quad \delta)\nabla v(y,0) \\ &\quad + \tfrac{1}{2}(\Delta y \quad \delta) H_v(y,0)\begin{pmatrix}\Delta y \\ \delta\end{pmatrix} + o(\Delta y^2 + \delta^2). \end{aligned}$$

Theorem (Mostovyi, S.)

(i) *The values of quadratic optimizations*

$$\begin{pmatrix} a(x,x) & 0 \\ a(x,d) & -\frac{x}{y} \end{pmatrix} \begin{pmatrix} b(y,y) & 0 \\ b(y,d) & -\frac{y}{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\frac{y}{x} a(d,d) + \frac{x}{y} b(d,d) = a(x,d) b(y,d).$$

(ii) *The optimizers of the quadratic problems are related*

$$U''(\widehat{X}_T(x,0))\widehat{X}^0_T(x,0) \begin{pmatrix} M^0_T(x,0)+1 \\ M^1_T(x,0)+xF \end{pmatrix} = - \begin{pmatrix} a(x,x) & 0 \\ a(x,d) & -\frac{x}{y} \end{pmatrix} \widehat{Y}^0_T(y,0) \begin{pmatrix} N^0_T(y,0)+1 \\ N^1_T(y,0)-yF \end{pmatrix},$$

$$V''(\widehat{Y}_T(y,0))\widehat{Y}_T(y,0) \begin{pmatrix} 1+N^0_T(y,0) \\ -yF+N^1_T(y,0) \end{pmatrix} = \begin{pmatrix} b(y,y) & 0 \\ b(y,d) & -\frac{y}{x} \end{pmatrix} \widehat{X}_T(x,0) \begin{pmatrix} 1+M^0_T(x,0) \\ xF+M^1_T(x,0) \end{pmatrix}.$$

(iii) *The product of any of* $\widehat{X}(x,0)$, $\widehat{X}(x,0)M^0(x,0)$, $\widehat{X}(x,0)M^1(x,0)$ *and any of* $\widehat{Y}(y,0)$, $\widehat{Y}(y,0)N^0(y,0)$, $\widehat{Y}(y,0)N^1(y,0)$ *is a* $\mathbb{P}$*-martingale.*

Derivatives of the optimizers

Theorem (Mostovyi, S.)

Let us set

$$X'_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(1+M^0_T(x,0)), \quad Y'_T(x,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(1+N^0_T(y,0)),$$

and

$$X^d_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(M^1_T(x,0)+xF), \quad Y^d_T(y,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(N^1_T(y,0)-yF).$$

Then, we have

$$\lim_{|\Delta x|+|\delta|\to 0} \frac{1}{|\Delta x|+|\delta|}\left|\widehat{X}_T(x+\Delta x,\delta)-\widehat{X}_T(x,0)-\Delta x X'_T(x,0)-\delta X^d_T(x,0)\right| = 0,$$

$$\lim_{|\Delta y|+|\delta|\to 0} \frac{1}{|\Delta y|+|\delta|}\left|\widehat{Y}_T(y+\Delta y,\delta)-\widehat{Y}_T(y,0)-\Delta y Y'_T(y,0)-\delta Y^d_T(y,0)\right| = 0,$$

where the convergence takes place in $\mathbb{P}$-probability.

Approximation of the optimal trading strategies

Observation: because the "random endowment" is multiplicative, proportions work better.
With

$$M^R = S^0 - \widehat{\pi}(x,0) \cdot \langle M \rangle,$$

let

$$\gamma^0 \cdot M^R = \frac{M^0(x,0)}{x} \quad \textit{and} \quad \gamma^1 \cdot M^R = \frac{M^1(x,0)}{x},$$

and

$$\begin{aligned} \sigma_\varepsilon &\triangleq \inf\left\{t \in [0,T] : \ |M^0_t(x,0)| \geq \tfrac{x}{\varepsilon} \ \textit{ or } \ \langle M^0(x,0)\rangle_t \geq \tfrac{x}{\varepsilon}\right\}, \\ \tau_\varepsilon &\triangleq \inf\left\{t \in [0,T] : \ |M^1_t(x,0)| \geq \tfrac{x}{\varepsilon} \ \textit{ or } \ \langle M^1(x,0)\rangle_t \geq \tfrac{x}{\varepsilon}\right\}, \\ & \qquad \varepsilon > 0, \end{aligned}$$

as well as

$$\gamma^{0,\varepsilon} = \gamma^0 1_{\{[0,\sigma_\varepsilon]\}} \quad \textit{and} \quad \gamma^{1,\varepsilon} = \gamma^1 1_{\{[0,\tau_\varepsilon]\}}, \quad \varepsilon > 0.$$

Approximation of the optimal trading strategies

Let us set

$$\begin{aligned} dX_t^{\Delta x,\delta,\varepsilon} &= X_t^{\Delta x,\delta,\varepsilon}(\widehat{\pi}_t(x,0) + \Delta x \gamma_t^{0,\varepsilon} + \delta(\nu_t + \gamma_t^{1,\varepsilon}))dS_t^{\delta}, \\ X_0^{\Delta x,\delta,\varepsilon} &= x + \Delta x. \end{aligned}$$

Note that

$$X^{\Delta x,\delta,\varepsilon} = (x+\Delta x)\frac{\widehat{X}(x,0)}{x}\frac{\mathcal{E}\left((\Delta x \gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\cdot M^R\right)}{\mathcal{E}(-\delta\nu\cdot S^0)}.$$

Theorem (Mostovyi, S.)

There exists a function $\varepsilon = \varepsilon(\Delta x, \delta)$*, such that*

$$\mathbb{E}\left[U\left(X_T^{\Delta x,\delta,\varepsilon(\Delta x,\delta)}\right)\right] = u(x+\Delta x,\delta) - o(\Delta x^2 + \delta^2).$$

Risk-tolerance wealth process

Definition

For $x > 0$ and $\delta \in \mathbb{R}$, the risk-tolerance wealth process is a maximal wealth process $R(x, \delta)$, such that

$$R_T(x, \delta) = -\frac{U'(\widehat{X}_T(x, \delta))}{U''(\widehat{X}_T(x, \delta))}.$$

Remark

This process was introduced in Kramkov and S. (2006) in the context of asymptotic analysis of utility-based prices.

Theorem (Kramkov and S. (2006))

The following assertions are equivalent:

(1) *The risk-tolerance wealth process* $R(x,0)$ *exists.*

(2) *The value function* u *admits the expansion quadratic expansion at* $(x,0)$ *and* $u_{xx}(x,0) = -\frac{y}{x}a(x,x)$ *satisfies*

$$\frac{(u_x(x,0))^2}{u_{xx}(x,0)} = \mathbb{E}\left[\frac{\left(U'(\widehat{X}_T(x,0)\right)^2}{U''(\widehat{X}_T(x,0))}\right],$$

$$u_{xx}(x,0) = \mathbb{E}\left[U''(\widehat{X}_T(x,0)\left(\frac{R_T(x,0)}{R_0(x,0)}\right)^2\right].$$

(3) *The value function* v *admits the quadratic expansion at* $(y,0)$ *and* $v_{yy}(y,0) = \frac{x}{y}b(y,y)$ *satisfies*

$$y^2 v_{yy}(y,0) = \mathbb{E}\left[\left(\widehat{Y}_T(y,0)\right)^2 V''(\widehat{Y}_T(y,0))\right] = xy\mathbb{E}^{\mathbb{R}(x,0)}\left[B(\widehat{Y}_T(y,0))\right].$$

Theorem (..Continued)

In addition, if these assertions are valid, then the initial value of $R(x)$ *is given by*

$$R_0(x,0) = -\frac{u_x(x,0)}{u_{xx}(x,0)} = \frac{x}{a(x,x)},$$

the product $R(x,0)Y(y,0) = (R_t(x,0)Y_t(y,0))_{t\in[0,T]}$ *is a uniformly integrable martingale and*

$$\lim_{\Delta x\to 0} \frac{\widehat{X}_T(x+\Delta x,0) - \widehat{X}_T(x,0)}{\Delta x} = \frac{R_T(x,0)}{R_0(x,0)},$$

$$\lim_{\Delta y\to 0} \frac{\widehat{Y}_T(y+\Delta y,0) - \widehat{Y}_T(y,0)}{\Delta y} = \frac{\widehat{Y}_T(y,0)}{y},$$

where the limits take place in $\mathbb{P}$*-probability.*

For $x > 0$ and with $y = u_x(x, 0)$, let us define

$$\frac{d\widetilde{\mathbb{R}}(x,0)}{d\mathbb{P}} \triangleq \frac{R_T(x,0)\widehat{Y}_T(y,0)}{R_0(x,0)y},$$

and choose $\frac{R(x,0)}{R_0(x,0)}$ as a numéraire, i.e., let us set

$$S^{R(x,0)} \triangleq \left(\frac{R_0(x,0)}{R(x,0)}, \frac{R_0(x,0)S}{R(x,0)} \right).$$

We define the spaces of martingales

$$\widetilde{\mathcal{M}}^2(x,0) \triangleq \left\{ M \in \mathbf{H}_0^2(\widetilde{\mathbb{R}}(x,0)) : \; M = H \cdot S^{R(x,0)} \right\},$$

and $\widetilde{\mathcal{N}}^2(y,0)$ it the orthogonal complement in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x,0))$.

Risk-tolerance wealth process and a Kunita-Watanabe decomposition

Theorem (Mostovyi, S.)

Let us assume that the risk-tolerance process $R(x,0)$ exists. Consider the Kunita-Watanabe decomposition of the square integrable martingale

$$P_t \triangleq \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\left[\left(A(\widehat{X}_T(x,0)) - 1\right) xF|\mathcal{F}_t\right], \quad t \in [0,T].$$

given by

$$P = P_0 - \widetilde{M}^1 - \widetilde{N}^1, \quad \textit{where} \quad \widetilde{M}^1 \in \widetilde{\mathcal{M}}^2(x,0), \quad \widetilde{N}^1 \in \widetilde{\mathcal{N}}^2(y,0), \quad P_0 \in \mathbb{R}.$$

Theorem (..Continued)

Then, the optimal solutions $M^1(x,0)$ *and* $N^1(y,0)$ *of the auxiliary quadratic optimization problems for* $u_{\delta\delta}$ *and* $v_{\delta\delta}$ *can be obtained from the Kunita-Watanabe decomposition (above) by reverting to the original numéraire, through the identities:*

$$\widetilde{M}^1_t = \frac{\widehat{X}_t(x,0)}{R_t(x,0)} M^1_t(x,0), \quad \widetilde{N}^1_t = \frac{x}{y} N^1_t(y,0), \quad t \in [0,T].$$

In addition, the Hessian terms in the quadratic expansion of u *and* v *can be identified as*

$$\begin{aligned} a(d,d) &= \frac{R_0(x,0)}{x} \inf_{\widetilde{M} \in \widetilde{\mathcal{M}}^2(x,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)} \left[\left(\widetilde{M}_T + xF\left(A\left(\widehat{X}_T(x,0) \right) - 1 \right) \right)^2 \right] + C_a. \\ &= \frac{R_0(x,0)}{x} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)} \left[\left(\widetilde{N}^1_T \right)^2 \right] + \frac{R_0(x,0)}{x} P_0^2 + C_a, \end{aligned}$$

where $C_a \triangleq x^2 \mathbb{E}^{\mathbb{R}(x,0)} \left[F^2 \frac{A(\widehat{X}_T(x,0))-1}{A(\widehat{X}_T(x,0))} - G \right].$

Theorem (..Continued)

$$b(d,d) = \frac{R_0(x,0)}{x} \inf_{\widetilde{N}\in\mathcal{N}^2(y,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\left[\left(\widetilde{N}_T + yF\left(A\left(\widehat{X}_T(x,0)\right)-1\right)\right)^2\right] + C_b.$$
$$= \frac{R_0(x,0)}{x}\left(\frac{y}{x}\right)^2 \mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\left[\left(\widetilde{M}_T^1\right)^2\right] + \frac{R_0(x,0)}{x}\left(\frac{y}{x}\right)^2 P_0^2 + C_b,$$

where $C_b \triangleq y^2\mathbb{E}^{\mathbb{R}(x,0)}\left[G + F^2\left(1 - A\left(\widehat{X}_T(x,0)\right)\right)\right]$. *The cross terms in the Hessians of* u *and* v *are identified as*

$$a(x,d) = P_0$$

and $b(y,d)$ *is given by*

$$b(y,d) = \frac{y}{x}\frac{P_0}{a(x,x)}.$$

With these identifications, all the expansions of the value functions above hold.

Summary

- look at the simultaneous perturbations of the market price of risk and the initial wealth
- formulate quadratic optimization problems and relate the second-order approximations of both primal and dual value functions to these problems.
- in case when the risk-tolerance wealth process exists, we used it as a numéraire, and changed the measure accordingly, to identify solutions to the quadratic optimization problems above in terms of a Kunita-Watanabe decomposition.