

# Rough volatility, market impact and no-arbitrage

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# A universal law to explain

## What we know from data :

- Volatility is rough !
- This is almost universally true. . .

## What we want to understand :

- Why is volatility rough ?
- Something universal in finance → should be related to some no arbitrage concept. Can we make this link ?
- We will use various results from econophysics, notably obtained by T. Jaisson.

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# Market impact

## Some definitions

- Market impact is the link between the volume of an order (either market order or metaorder) and the price moves during and after the execution of this order.
- We focus here on the impact function of metaorders, which is the expectation of the price move with respect to time during and after the execution of the metaorder.
- We call permanent market impact of a metaorder the limit in time of the impact function (that is the average price move between the start of the metaorder and a long time after its execution).

# Market impact

## Two possible views the impact

- Market impact as a way to pass on private information to the price : large investors react to information signals on the future expectation of the price using metaorders. In such approaches, metaorders reveal fundamental price moves but do not really cause them. In particular, if a metaorder is executed for no reason, it should not have any long term impact on the price.
- Mechanical view : A metaorder moves the price through its volume, whether it is informed or not.
- We take the second viewpoint.

# Market impact

## Linear permanent impact

- Let  $P_t$  be the asset price at time  $t$ . Consider a metaorder with total volume  $V$ .



$$PMI(V) = \lim_{s \rightarrow +\infty} \mathbb{E}[P_s - P_0 | V].$$

- Price manipulation is a roundtrip with negative average cost.
- From Huberman and Stanzl and Gatheral : Only linear permanent market impact can prevent price manipulation :  
 $PMI(V) = kV$ .

# Market impact

## CAPM like argument for linear permanent impact

- $n$  investors in the market. Two dates :  $t = 0$  and  $t = 1$ .
- $N$  shares spread between the agents, price  $P$  for the asset.
- Every investor  $i$  estimates that the law of the price at time 1 has expectation  $E_i$  and variance  $\Sigma_i$ . He chooses his number of asset  $N_i$  such that

$$N_i = \operatorname{argmax}_x [x(E_i - P) - \lambda_i x^2 \Sigma_i].$$

- We get

$$N_i = \frac{E_i - P}{2\lambda_i \Sigma_i}.$$



# Market impact

## CAPM like argument for linear permanent impact

- Since  $\sum_{i=1}^n N_i = N$ , we deduce

$$P = \frac{\sum_{i=1}^n \frac{E_i}{2\lambda_i \Sigma_i} - N}{\sum_{i=1}^n \frac{1}{2\lambda_i \Sigma_i}}.$$

- Let us now assume that the total number of shares becomes  $N - N_0$  due to the action of some non-optimizing agent needing to buy some shares (for cash flow reasons for example). The new indifference price is

$$P^+ = P + \frac{N_0}{\sum_{i=1}^n \frac{1}{2\lambda_i \Sigma_i}} = P + kN_0.$$

# Dynamics

## Assumptions

- All market orders are part of metaorders.
- Let  $[0, S]$  be the time during which metaorders are being executed (which can be thought of as the trading day). Let  $v_i^a$  (resp.  $v_i^b$ ) be the volume of the  $i$ -th sell (resp. buy) metaorder and  $N_S^a$  (resp.  $N_S^b$ ) be the number of sell (resp. buy) metaorders up to time  $S$ . Finally, write  $V_S^a$  and  $V_S^b$  for cumulated sell and buy order flows up to time  $S$ .
- We assume

$$P_S = P_0 + k \left( \sum_{i=1}^{N_S^a} v_i^a - \sum_{i=1}^{N_S^b} v_i^b \right) + Z_S = P_0 + k(V_S^a - V_S^b) + Z_S,$$

with  $Z$  a martingale term that we neglect.

# Dynamics

## Martingale assumption

- We furthermore assume that the price  $P_t$  is a martingale. We obtain

$$P_t = P_0 + \mathbb{E}[k(V_S^a - V_S^b)|F_t].$$

- We suppose that  $\lim_{S \rightarrow +\infty} \mathbb{E}[k(V_S^a - V_S^b)|F_t]$  is well defined.

This means

$$\mathbb{E}[(V_{S+h}^a - V_{S+h}^b) - (V_S^a - V_S^b)|F_t] \rightarrow 0,$$

that is the order flow imbalance between  $S$  and  $S + h$  is asymptotically (in  $S$ ) not predictable at time  $t$ .

# Dynamics

## Price dynamics

- Under the preceding assumptions, we finally get

$$P_t = P_0 + k \lim_{S \rightarrow +\infty} \mathbb{E}[(V_S^a - V_S^b) | F_t].$$

- Martingale price.
- Linear permanent impact, independent of execution mode.
- The price process only depends on the global market order flow and not on the individual executions of metaorders. We thus do not need to assume that the market sees the execution of metaorders as it is usually done.
- Market orders move the price because they change the anticipation that market makers have about the future of the order flow.

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# Hawkes specification

## Hawkes propagator

- We now assume that buy and sell order flows are modeled by independent Hawkes processes with same parameters  $\mu$  and  $\phi$ . All orders have same volume  $v$ .
- In this case, the general equation above rewrites as the following propagator dynamic

$$P_t = P_0 + \int_0^t \zeta(t-s)(dN_s^a - dN_s^b),$$

with  $\zeta(t) = kv(1 + \int_t^{+\infty} \psi(u) - \int_0^t \psi(u-s)\phi(s)dsdu)$ .

- The propagator kernel compensates the correlation of the order flow implied by the Hawkes dynamics to recover a martingale price. Note that the kernel does not tend to 0 since there is permanent impact.

# Adding our own transactions

## Labeled order

- In the above framework,  $N^a$  and  $N^b$  are the flows of anonymous market orders.
- Now assume we arrive on the market, executing our own (buy) metaorder. Our flow is a Poisson process  $P^{F,\tau}$  with intensity  $F$  between  $t = 0$  and  $t = \tau$ .
- According to the propagator approach, we get

$$P_t = P_0 + \int_0^t \zeta(t-s)(dN_s^a - dN_s^b) + \int_0^t \zeta(t-s)dP_s^{F,\tau}.$$

# Impact function

## Square root law

- We get that the impact function of a metaorder executed between 0 and  $\tau$  is

$$MI(t) := \mathbb{E}[P_t - P_0] = F \int_0^{t \wedge \tau} \zeta(t-s) ds.$$

- In particular, the permanent impact of this metaorder is  $F \lim_{t \rightarrow +\infty} \zeta(t)$ .
- From Bouchaud *et al.*, the shape of the impact function MI during the execution of a metaorder must be close to square root to ensure price diffusivity. See also Pohl *et al.*
- In the Hawkes framework, this can be compatible with linear permanent impact !



# Renormalizing impact

## Scaling

- In fact, one considers an asymptotic regime where the length of metaorders  $\tau^T$  tends to infinity.
- We rescale the impact function in time over  $[0, 1]$  and multiply it by a proper factor in space. That is we consider

$$RMI^T(t) = b_T MI(t\tau^T).$$

- This is exactly what people have done empirically when computing impact curves mixing various types of metaorders.
- We want to obtain that  $RMI^T(t)$  behaves at  $t^{1-\alpha}$ .

# Renormalizing impact

## Scaling

- Only one subtle specification of the Hawkes parameters can lead to the target function  $t^{1-\alpha}$  :

$$\|\phi\|_1 \rightarrow 1, \phi(x) \underset{x \rightarrow \infty}{\sim} K/x^{1+\alpha}, \tau^T(1 - \|\phi\|_1)^{1/\alpha} \rightarrow 0.$$

- Square root law  $\rightarrow \alpha = 1/2$ .
- In term of rough volatility models, this corresponds to  $H = \alpha - 1/2 = 0$ ! Indeed,

$$N_t \approx \int_0^t \sigma_s^2 ds$$

and we have the following result :

# Non-degenerate limit for nearly unstable Hawkes processes

## Rough Heston model

For  $\alpha > 1/2$ , the sequence of renormalized Hawkes processes converges to some process which is differentiable on  $[0, 1]$ .

Moreover, the law of its derivative  $V$  satisfies

$$V_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda (1-V_s) ds + \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\lambda}{\mu^*}} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s.$$

Now recall Mandelbrot-van-Ness representation :

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left( \frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

Therefore we have a rough Heston model with  $H = \alpha - 1/2$ .

# Summary

## From no-arbitrage to volatility

- We made two assumptions : Linear permanent impact and price diffusivity (square root law).
- Only modeling assumption : Hawkes dynamics for the order flow (reasonable...).
- This leads to rough volatility with  $H = 0$ .

# The two no-arbitrage indexes

## Our intuition

- No “strong” arbitrage  $\rightarrow H \approx 1/2$  for the stock price.
- No “statistical” arbitrage  $\rightarrow H \approx 0$  for the volatility.

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# The case $H = 0$

## Limiting case for fractional Brownian motion

- Let

$$X_t^H = \frac{B_t^H - \int_0^1 B_s^H ds}{\sqrt{H}}.$$

- We have

$$\lim_{H \rightarrow 0} X_t^H = X_t,$$

with

$$\begin{aligned} \mathbb{E}[X_s X_t] &= -\log|t - s| \\ &+ \int_0^1 \log|t - u| du + \int_0^1 \log|s - v| dv - \int_0^1 \int_0^1 \log|v - u| dv du. \end{aligned}$$

# The case $H = 0$

## Comments

- Warning : not a usual convergence. We consider generalized Gaussian processes viewed as Gaussian measures on the space  $\mathcal{S}'(\mathbb{R})/\mathbb{R}$ .
- The limiting object is somehow degenerate (white noise type behavior).
- Related to fractal processes : Mandelbrot, Kahane, Bacry, Muzy...