Mean Field Games with Branching

Zhenjie Ren
Joint work with *Julien Claisse* and *Xiaolu Tan*

CEREMADE, Université Paris-Dauphine

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What is Mean Field Equilibrium?

A deterministic measure flow $(p_t)_{t \le T}$ is a MFE if

$$\begin{cases} a^* = \text{argmax}_a \mathbb{E}\Big[\int_0^T f(t, X_t^a, p_t, a_t)dt + g(X_T^a, p_T) \Big] \\ p_t = \mathcal{L}(X_t^{a^*}) \end{cases}$$

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Quite rich literature...

Something missing in the previous story...

In each n-player game, the number of players is fixed. Therefore, as the limit of n-player equilibrium, the MFE also considers a population of constant size. It can be a major constraint when we apply MFG to economy (demography), biology (prey-predator) or finance (insurance).

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Introduce the MFG with branching !

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- *Immigration* ...

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$$a^{i,*} = \text{argmax}_a \mathbb{E}\Big[\int_{s_i}^{\tau_i \wedge T} \big(f(t, X_t^i, \mu_t) - \tfrac{1}{2} a_t^2\big) dt + g(X^i_{\tau_i \wedge T}, \mu_{\tau_i \wedge T}) \Big]$$

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where s_i is the birthday of Player i.

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Let $n \to \infty$. We have

$$\mu_t^n \longrightarrow \frac{\mathbb{E}\big[\sum_{i \in V_t} \delta_{X_t^i} \big]}{\mathbb{E}\big[\# V_t \big]} =: p_t \quad \text{is a probability measure.}$$

MFE and PDE characterization

A deterministic measure flow $(p_t)_{t\leq T}$ is a MFE if

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Similar to the classic MFE, it can be characterized by the PDE system

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A probability law of a branching tree P is a MFE if

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⋆ we prove the mapping $P \mapsto P$ is u.h.c. and has compact range, and use Schauder's fixed point theorem to prove the existence of MFE.

Thank you for your attention!