Affine Volterra processes and models for rough volatility

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Rough volatility models

- Empirical studies indicate **volatility is rougher than BM**: Gatheral, Jaisson & Rosenbaum ('14); Bennedsen, Lunde, Pakkanen ('16), ...
- Subsequent development of **stochastic models with this feature**: Gatheral, Jaisson & Rosenbaum ('14); Guennoun, Jacquier & Roome ('14); Bayer, Friz & Gatheral (15); Bennedsen, Lunde, Pakkanen ('16); **El Euch & Rosenbaum ('16,'17)**, ...
- Further literature: **sites.google.com/site/roughvol/home/risks-1**
- These models are able to
  - match roughness of time series data
  - fit implied volatility skew remarkably well
  - admit in some cases microstructural justification

Heston model

$$\frac{dS_t}{S_t} = \sqrt{X_t} d\widetilde{W}_t$$

$$X_t = X_0 + \int_0^t \left(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\right)$$

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Theorem (Heston, '93). Let $S_0 = 1$. Fix $u \in \mathrm{i}\mathbb{R}$. Assume ψ solves the **Riccati equation**

$$\psi' = \frac{1}{2}(u^2 - u) - (u\rho\sigma - \kappa)\psi + \frac{\sigma^2}{2}\psi^2, \qquad \psi(0) = 0,$$

and define $\phi(T) = \int_0^T \kappa\theta\psi(t)dt$. Then

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T)+\psi(T)X_0}$$

Rough Heston model of El Euch & Rosenbaum ('16)

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$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

with $\alpha \in (\frac{1}{2}, 1)$.

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- Similar to Riemann–Liouville fractional Brownian. Lévy ('53).
- Hölder continuous paths of any order less than $H = \alpha - \frac{1}{2}$
- Microstructural foundation as scaling limit of Hawkes processes

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But:

- Existence and uniqueness is non-trivial.
- Not a semimartingale, not Markovian ...
- ... not clear how to usefully describe its law.

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with $\alpha \in (\frac{1}{2}, 1)$. **Notation:** $D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds$

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Theorem (El Euch & Rosenbaum, '16). Let $S_0 = 1$. Fix $u \in \mathrm{i}\mathbb{R}$. Assume ψ solves the **fractional Riccati equation**

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and define $\phi(T) = \int_0^T \kappa\theta\chi(t)dt$ and $\chi(T) = \int_0^T D^\alpha\psi(t)dt$. Then

$$\mathbb{E}[e^{u\log S_T}] = e^{\phi(T) + \chi(T)X_0}$$

Why?

Characteristic function of affine diffusions

Consider an affine diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with $b(x)$ and $a(x) = \sigma(x)^2$ affine in x. How to derive its c.f.?

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(i) Define $M_t = e^{\phi(T-t)+\psi(T-t)X_t}$ with ϕ, ψ from Riccati.

(ii) Itô and Riccati imply M is local martingale. $M_T = e^{uX_T}$.

(iii) If true martingale, then $\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{\phi(T-t)+\psi(T-t)X_t}$.

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$$\log \mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s)^2 a(\mathbb{E}[uX_s \mid \mathcal{F}_t])ds.$$

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Here there is hope.

Affine Volterra processes

A continuous E-valued solution X of the stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

is called an **affine Volterra process** (of convolution type). Data:

- State space $E \subseteq \mathbb{R}^d$ and initial condition $X_0 \in E$.
- Affine diffusion and drift coefficients

$$a(x) = A^0 + A^1 x_1 + \cdots + A^d x_d$$
$$b(x) = b^0 + b^1 x_1 + \cdots + b^d x_d$$

  with $A^i \in \mathbb{S}^d$, $b^i \in \mathbb{R}^d$, and $a(x) = \sigma(x)\sigma(x)^\top$ for all $x \in E$.
- Matrix-valued kernel $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$.

Affine Volterra processes

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

- **Example:** $K(t) \equiv \mathrm{id}$ gives standard affine diffusions.
- **Example:** The rough CIR process of Rosenbaum & El Euch uses

$$K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$$

- **Example:** The full rough Heston model uses $d = 2$ and

$$K(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \end{pmatrix}$$

Conditional characteristic function

Theorem (∗). Fix a row vector $u \in (\mathbb{C}^d)^*$. Assume the function $\psi \in L^2_{\text{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$ solves the **Riccati–Volterra equation**

$$\psi = uK + \left(\psi B + \frac{1}{2}A(\psi)\right) * K$$

where $A(\psi) = (\psi A^1 \psi^\top, \ldots, \psi A^d \psi^\top)$. Fix $T < \infty$ and define

$$Y_t = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s) a(\mathbb{E}[X_s \mid \mathcal{F}_t]) \psi(T-s)^\top ds.$$

Then $\{\exp(Y_t),\, 0 \le t \le T\}$ is a local martingale and, if it is a true martingale, one has

$$\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{Y_t}, \quad t \le T.$$

Conditional expectations

- Take expectations in

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

to obtain

$$\mathbb{E}[X] = X_0 + (K * 1)\, b^0 + (KB) * \mathbb{E}[X]$$

- Get $\mathbb{E}[X_t]$ by variation of constants formula using resolvent of KB.
- Conditional expectations are similar.

Concrete specifications

Given a specification of E, $K(t)$, $a(x)$, $b(x)$, three things need proof:

- Existence of X (hence uniqueness of ψ)
- Existence of ψ (hence uniqueness of X)
- Martingale condition

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We do this for three classes of specifications:

- Volterra Ornstein–Uhlenbeck: $E = \mathbb{R}^d$
- Volterra square-root: $E = \mathbb{R}^d_+$
- Volterra Heston: $E = \mathbb{R} \times \mathbb{R}_+$

Volterra Heston model

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- **Dynamics** with $\kappa\theta \geq 0$, $\sigma \geq 0$, $d\langle \widetilde{W}, W \rangle_t = \rho\, dt$:

$$\frac{dS_t}{S_t} = \sqrt{V_t}\, d\widetilde{W}_t$$

$$V_t = V_0 + \int_0^t K(t-s) \left(\kappa(\theta - V_s) ds + \sigma \sqrt{V_s}\, dW_s \right)$$

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- **Riccati–Volterra equation:**

$$\psi_1 = u_1$$

$$\psi_2 = u_2 K + K * \left(\frac{1}{2} \left(u_1^2 - u_1 \right) + (\rho\sigma u_1 - \kappa)\psi_2 + \frac{1}{2}\sigma^2 \psi_2^2 \right)$$

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- **Assumption:** K is **completely monotone** and

$$t \mapsto t^\gamma K(t) \text{ is locally Lipschitz on } [0, \infty)$$

for some $\gamma < 1/2$.

Volterra Heston model

Theorem.

- The stochastic Volterra equation has a unique in law $\mathbb{R} \times \mathbb{R}_+$-valued continuous weak solution $(\log S, V)$ for any initial condition $(\log S_0, V_0) \in \mathbb{R} \times \mathbb{R}_+$. The paths of V are Hölder continuous of any order less than $H = 1/2 - \gamma$.
- For any $u \in (\mathbb{C}^2)^*$ such that

$$\operatorname{Re} u_1 \in [0,1] \text{ and } \operatorname{Re} u_2 \leq 0,$$

  the Riccati–Volterra equation has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^*)^2)$, which satisfies $\operatorname{Re} \psi_2 \leq 0$.
- The martingale condition in Theorem (∗) holds, as does the affine transform formula.
- The process S is a martingale.

Why complete monotonicity?

- A **resolvent of the first kind** of K is a kernel L such that

$$K * L = L * K \equiv \mathrm{id}$$

In general L is a measure, for instance $L(dt) = \delta_0(dt)$ if $K \equiv \mathrm{id}$.

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- **Example:** If K is **completely monotone**, then L exists and is the sum of a point mass in zero and a completely monotone function.

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Shift operator: $\Delta_h f(t) = f(t+h)$

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Lemma. Setting of Theorem (∗). Assume K has resolvent of the first kind L. Define

$$\pi_h = \Delta_h \psi * L - \Delta_h(\psi * L),$$

and assume $\pi_h \in BV_{\mathrm{loc}}(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ for $h = T - t$. Then

$$Y_t = \phi(h) + (\Delta_h \psi * L)(0) X_t - \pi_h(t) X_0 + (d\pi_h * X)_t$$

with $h = T - t$ and $\phi(h) = \int_0^h \left(\psi(s) b^0 + \frac{1}{2}\psi(s) A^0 \psi(s)^\top\right) ds$.

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In particular: For $E = \mathbb{R}^d_+$, verify martingale condition by controlling the signs of the real parts of $(\Delta_h \psi * L)(0)$, $\pi_h(t)$, and $d\pi_h$.

Why complete monotonicity?

Moreover: Fourier–Laplace transform exponential-affine in $\{X_s \colon s \leq t\}$,

$$\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = \exp\Big(\phi(h) + (\Delta_h \psi * L)(0) X_t - \pi_h(t) X_0 + (d\pi_h * X)_t\Big)$$

Conclusion

- Affine Volterra processes admit affine transform formulas despite lack of semimartingale and Markov properties
- Tools from deterministic theory of Volterra equations, e.g. resolvents of first and second kind. Read the spectacular book *Volterra integral and functional equations*, 1990, by Gripenberg, Londen, Staffans.
- Unknown or in progress :
  - Regularity of ψ (currently L^2_{loc})
  - Pathwise uniqueness for X (currently uniqueness in law)
  - Numerical methods for ψ
  - Numerical methods for X
  - Boundary attainment for Volterra square-root processes
  - Non-convolution kernels $K(t,s)$
  - Stationary case $X_t = \int_{-\infty}^{t}(\cdots)$
  - Etc.