Perfect hedging under endogenous permanent market impacts

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Motivation

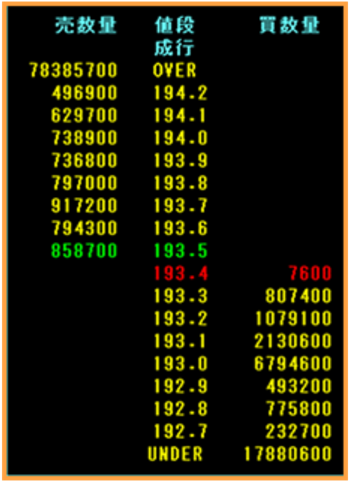
- Itô integral as Profit & Loss :

$$\int_0^T H_t \mathrm{d}S_t \approx \sum_{i=1}^{n} H_{t_{i-1}}(S_{t_i} - S_{t_{i-1}})$$

(buy $H_{t_{i-1}}$ units of S at t_{i-1} and sell them at t_i).

- The P&L is linear in the strategy H.
- The price process S is exogenously modeled.
- Reality ?

Limit order book



売数量	値段 成行	買数量
78385700	OVER	
496900	194.2	
629700	194.1	
738900	194.0	
736800	193.9	
797000	193.8	
917200	193.7	
794300	193.6	
858700	193.5	
	193.4	7600
	193.3	807400
	193.2	1079100
	193.1	2130600
	193.0	6794600
	192.9	493200
	192.8	775800
	192.7	232700
	UNDER	17880600

Price is nonlinear !

Nonlinear Profit & Loss

1. Denote by Θ the value of a security at future time T.
2. Let Y be a simple predictable process with $Y_0 = 0$, representing a trading strategy (holding Y_t units at time t).
3. Denote by $P_t(y)$ the market quote (price) for the quantity y of the security at time t.
4. The P&L = terminal value - total amount of money paid :

$$I(Y) = Y_T \Theta - \sum_{0 \leq t < T} P_t(\Delta Y_t)$$

5. If $P_t(y) = yS_t$ for a semimartingale S with $S_T = \Theta$, then

$$I(Y) = Y_T S_T - \sum_{0 \leq t < T} \Delta Y_t S_t = \int_0^T Y_t \mathrm{d} S_t$$

by integration-by-parts.

How to think about a market ?

Market is not big enough ...

- You (large trader) are not a price taker anymore.
- Price is nonlinear due to market (price) impact.

Market is still big ...

- The market doesn't care of your utility and your endowment (your payoff to hedge), to reach an equilibrium.
- You are a noise trader.
- The Bertrand competition among the liquidity suppliers
  - Bank and Kramkov (2013,2014)
  - Utility-indifference pricing
- A representative agent with concave, cash-additive utility.
  - ex. exponential utility
  - ambiguity aversion
  - Horst et al (2010)

(Concave, cash-additive) g-expectation

$(\Omega, \mathcal{F}, P)$: a prob. space supporting a Brownian motion $\{W_t\}$. Let $\{\mathcal{F}_t\}$ be the augmentation of the natural filtration.

Let $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a progressively measurable map with $g(\omega, t, 0) = 0$ and $z \mapsto g(\omega, t, z)$ being convex.

A map $D_T \ni X \mapsto \Pi = \{\Pi_t(X)\}$ is called the g expectation if there exists $Z = Z(X)$ such that (Π, Z) solves the BSDE

$$X = \Pi_t + \int_t^T g(\cdot, s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T]$$

where $\mathcal{D}_T$ is a vector space of $\mathcal{F}_T$ random variables.

Ex. $\Pi_t = E[X|\mathcal{F}_t]$ is the g expectation with $g = 0$, $D_T = L^2(\mathcal{F}_T)$.

Utility

If (and essentially only if) $X \mapsto \Pi(X)$ is the g expectation, then

1. $\Pi_t(0) = 0$
2. $\Pi_t(X + X') = \Pi_t(X) + X'$ if X' is $\mathcal{F}_t$ measurable
3. $\Pi_t(\lambda X + (1 - \lambda)X') \geq 0$ if $\lambda \in [0, 1]$, $\Pi_t(X) \geq 0$, $\Pi_t(X') \geq 0$
4. $\Pi_t(X) \geq \Pi_t(X')$ if $\Pi_s(X) \geq \Pi_s(X')$ for some $s \geq t$

Π can be called time-consistent monetary utility function.
$-\Pi$ is a dynamic risk measure.

Consider a representative liquidity supplier ("market") to make a price quote $P_t(y)$ according to the utility indifference principle:

$$\begin{aligned} P_t(y) &= \inf\{p \in \mathbb{R}; \Pi_t(H_M + (Z_t - y)\Theta + p) \geq \Pi_t(H_M + Z_t\Theta)\} \\ &= \Pi_t(H_M + Z_t\Theta) - \Pi_t(H_M + (Z_t - y)\Theta) \end{aligned}$$

where H_M and Z_t are resp. the initial endowment and inventory.

Nonlinear price

Concave cash-additive utility indifference price $P_t(y) = P_t(Z_t, y)$,

$$P_t(z, y) := \Pi_t(H_M + z\Theta) - \Pi_t(H_M + (z - y)\Theta).$$

H_M and Θ are exogenous, $\mathcal{F}_T$ m'ble random variables.
Z_t : inventory of the security Θ at time t ($Z_0 = 0$).

Properties

- $y \mapsto P_t(z, y)$ is convex and increasing with $P_t(z, 0) = 0$
- In particular,

$$-P_t(z, -y) \leq P_t(z, y)$$

- zero round-trip-cost (no arbitrage) :

$$P_t(z, y) + P_t(z - y, -y) = 0$$

- nonlinear permanent market impact (cf. Guéant 2014)

Nonlinear stochastic integral

By clearing between you and the market, it holds $Z_t = -Y_t$ and so,

$$\begin{aligned} I(Y) &= Y_T\Theta - \sum_{0\le t<T} P_t(Z_t, \Delta Y_t) \\ &= Y_T\Theta - \sum_{0\le t<T} \Pi_t(H_M - Y_t\Theta) - \Pi_t(H_M - Y_{t+}\Theta) \\ &= Y_T\Theta - \sum_{j=1}^{n}(\Pi_{\tau_j}(H_M - Y_{\tau_j}\Theta) - \Pi_{\tau_j}(H_M - Y_{\tau_{j+1}}\Theta)) \\ &= H_M - \Pi_0(H_M) \\ &\qquad - \sum_{j=0}^{n}(\Pi_{\tau_{j+1}}(H_M - Y_{\tau_{j+1}}\Theta) - \Pi_{\tau_j}(H_M - Y_{\tau_{j+1}}\Theta)), \end{aligned}$$

where τ_j is the j-th transaction time, $\tau_0 = 0$ and $\tau_{n+1} = T$.
Earlier works: Kunita (1990), Bank and Kramkov (2013,2014)

Representation in terms of BSDE

Since

$$\Pi_{\tau_{j+1}}(H_M - y\Theta) - \Pi_{\tau_j}(H_M - y\Theta) = \int_{\tau_j}^{\tau_{j+1}} g(s, Z_s^y)\mathrm{d}s - \int_{\tau_j}^{\tau_{j+1}} Z_s^y \mathrm{d}W_s,$$

where Z^y is a part of the BSDE solution for $\Pi(H_M - y\Theta)$.

Lemma. Assume there exist a progressively measurable map

$$Z : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$$

and $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that $Z(\omega, t, y) = Z_t^y(\omega)$ for all $(\omega, t, y) \in \Omega_0 \times [0, T] \times \mathbb{R}$. Then,

$$I(Y) = H_M - \Pi_0(H_M) - \int_0^T g(s, Z_s^Y)\mathrm{d}s + \int_0^T Z_s^Y \mathrm{d}W_s,$$

where $Z_s^Y(\omega) = Z(\omega, s, Y_s(\omega))$.

Hedging Theorem

Extend the domain of the nonlinear integral $I(Y)$:

$$\mathcal{S} = \left\{ Y : \Omega \times [0,T] \to \mathbb{R};\ \text{adapted with } \int_0^T |Z_s^Y|^2 \mathrm{d}s < \infty \right\}.$$

Theorem. In addition to the assumption of Lemma, if there exist a progressively measurable map

$$Z^- : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$$

and $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that $Z(\omega, t, Z^-(\omega, t, z)) = z$ for all $(\omega, t, z) \in \Omega_0 \times [0,T] \times \mathbb{R}$. Then, for any $-H_L \in \mathcal{D}_T$, we have

$$-H_L = \Pi_0(H_M) - \Pi_0(H_M + H_L) + I(Y^*),$$

where $Y_t^*(\omega) = Z^-(\omega, t, Z_t(H_M + H_L)(\omega))$.

Proof

Since Π is the g expectation, there exists $Z^* := Z(H_M + H_L)$ s.t.

$$H_M + H_L = \Pi_0(H_M + H_L) + \int_0^T g(s, Z_s^*)\mathrm{d}s - \int_0^T Z_s^* \mathrm{d}W_s.$$

Define Y^* by $Y_t^*(\omega) = Z^-(\omega, t, Z_t^*(\omega))$. Then,

$$Z_t^{Y^*}(\omega) = Z(\omega, t, Y_t^*(\omega)) = Z_t^*(\omega).$$

Therefore by Lemma,

$$I(Y^*) = H_M - \Pi_0(H_M) - \int_0^T g(s, Z_s^*)\mathrm{d}s + \int_0^T Z_s^* \mathrm{d}W_s,$$

which implies the result.

Completeness condition

When does Z^- exist ?

- In the linear case, i.e., $g(t,z) = G_t z$, then

$$\Pi_t(X) = E^Q[X|\mathcal{F}_t], \quad \frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left\{\int_0^T G_t \mathrm{d}W_t - \frac{1}{2}\int_0^T G_t^2 \mathrm{d}t\right\}$$

and so, $P_t(y) = yS_t$ with $S_t = E^Q[\Theta|\mathcal{F}_t]$ and

$$Z_t^y = y\frac{\mathrm{d}}{\mathrm{d}t}\langle S, W\rangle_t.$$

This means that Z^- exists iff the volatility is positive.

- In the quadratic case, i.e., $g(t,z) = \gamma z^2/2$, $\gamma > 0$, we have

$$\Pi_t(X) = -\frac{1}{\gamma}\log E[\exp\{-\gamma X\}|\mathcal{F}_t]$$

with $\mathcal{D}_T := \{X \in L^1(\mathcal{F}_T); E[\exp\{a|X|\}] < \infty, \forall a > 0\}$, so ...

The case of exponential utilities

Proposition: Let $\Theta = s(W_T)$ and $H_M = h_M(W_T)$. Then,

1. The assumptions of Lemma and Theorem are satisfied if s and h_M are of linear growth and s is strictly monotone on $\mathbb{R}$.
2. If $s(w) = (w-k)_+$ with $k \in \mathbb{R}$, then

$$\lim_{y\to\infty} Z_t^y = \infty, \quad \lim_{y\to-\infty} Z_t^y = -\frac{\phi\left(\frac{k-W_t}{\sqrt{T-t}}\right)}{\gamma\sqrt{T-t}\Phi\left(\frac{k-W_t}{\sqrt{T-t}}\right)}.$$

   In particular Z^- does not exist.

Markov Lipschitz cases $\rightarrow$ Fukasawa and Stadje, to appear in FS.

Bachelier-type model

Let $\Theta = b + cW_T$, $c > 0$, $H_M = aW_T$ and $H_L = h_L(\Theta)$ (European payoff for a normally distributed asset).
Let $g(t,z) = \gamma z^2/2$ (an exponential utility). Then,

$$\begin{aligned}\Pi_t(H_M - y\Theta) &= -\frac{1}{\gamma}\log E[\exp\{-\gamma(a-y)(b+cW_T)\}|W_t] \\ &= (a-y)(b+cW_t) - \frac{T-t}{2}\gamma(a-y)^2c^2\end{aligned}$$

and so,

$$Z_t^y = -(a-y)c, \quad Z^-(\omega,t,z) = a + \frac{z}{c}.$$

The hedging strategy for $-h_L(\Theta)$ is

$$Y_t^* = a - \frac{1}{c}u(W_t,t),$$

where u is a solution of Burgers' equation $u_t + u_{xx}/2 = \gamma u u_x$.

Example : Put-like payoff

Suppose $\gamma > 0$, $a = 0$ and consider to hedge

$$2\lambda(K-\Theta)_+ \approx \lambda\left(K - \Theta + \frac{1}{\lambda\gamma}\log\cosh(-\lambda\gamma(K-\Theta))\right) =: -h_L(\Theta).$$

Since

$$h_L'(s) = \lambda(1 - \tanh(-\lambda\gamma(K-s))),$$

and the solution u of (backward) Burgers' equation

$$u_t(x,t) + \frac{1}{2}u_{xx}(x,t) = \gamma u u_x, \quad u(x,T) = ac + ch_L'(b+cx)$$

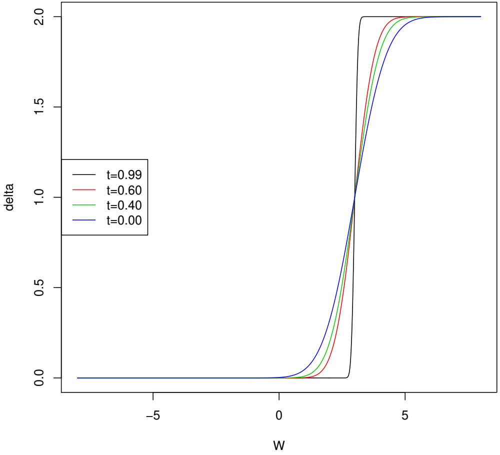
is given by

$$u(x,t) = \lambda c(1 - \tanh(\gamma\lambda c x + \gamma^2\lambda^2 c^2 t + \delta)),$$

where $\delta = \lambda\gamma(b-K) - \gamma^2\lambda^2 c^2 T$, the hedging strategy is

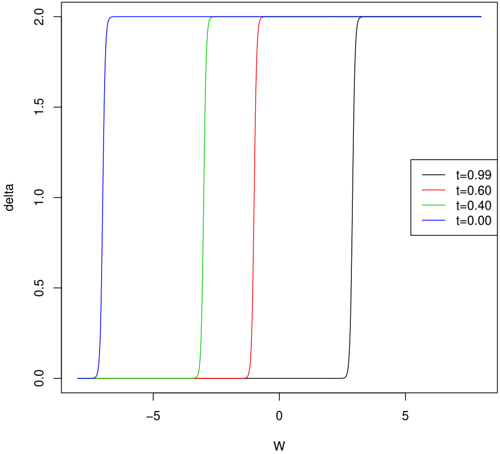
$$Y_t^* = -\lambda(1 - \tanh(\gamma\lambda(b + cW_t - K) - \gamma^2\lambda^2 c^2(T-t))).$$

The delta of a call option under the Bachelier model



A solution of the heat equation (diffusion).

Hedging strategy under $g(t, z) = \frac{\gamma}{2} z^2$



A solution of Burgers' equation (shockwave).

Comment : backward stochastic flow

The existence of Z^- (roughly) $\Leftrightarrow y \mapsto Z_t^y$ is a bijection for all t.

Recall $(\Pi(H_M - y\Theta), Z^y)$ is the solution of the BSDE with terminal condition $H_M - y\Theta$

In Markov cases, e.g., $H_M = h_M(W_T)$ and $\Theta = s(W_T)$, then a BSDE with terminal condition $h'_M(W_T) - ys'(W_T)$ has a solution $(Z^y, \hat{Z}^y)$. The comparison theorem is applicable if $s' > 0$ to show the monotonicity of Z^y in y.

Regularity: Ankirchner et al (2007)

Future work : the flow property of $y \mapsto Z^y$