Rough volatility from an affine point of view

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Rough volatility models

- Universal phenomenon discovered by Gatheral, Jaisson and Rosenbaum (2014): Volatility is rough
- Certain rough volatility models in the class of affine Volterra processes (Abi Jaber, Larsson, Pulido (2017)), can be obtained as scaling limits of Hakwes processes (see "The microstructural foundations of leverage effect and rough volatility" by El Euch, Fukasawa, Rosenbaum (2016))

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Goal of this work

- Unifying Markovian framework for Hawkes type processes and affine Volterra processes via (infinite dimensional) affine processes
- Novel (numerical) approximations of rough models via finite dimensional affine processes

Motivating examples: Hawkes process and rough Heston model

- A (one-dimensional) Hawkes process N is a process that jumps by 1 with intensity

$$\lambda_t = \lambda_0 + \int_0^t \varphi(t-s)b(s)ds + \int_0^t \varphi(t-s)dN_s.$$

with a locally integrable kernel φ and a deterministic function b.

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- The rough Heston model consists of a log-price Y and a instantaneous variance process X such that

$$Y_t = Y_0 - \frac{1}{2}\int_0^t X_s ds + \int_0^t \sqrt{X}_s dB_{s,2},$$
$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\kappa(\theta - X_s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sqrt{X}_s dB_{s,1},$$

where $\alpha = H + \frac{1}{2} \in (\frac{1}{2}, 1)$ and B_1 and B_2 are correlated Brownian motions.

Stochastic Volterra equations - Setting

- State space $E = C \times \mathbb{R}^n \subseteq \mathbb{R}^d$, with C a closed proper convex cone.
- The components of an E-valued process Z are denoted by $Z = (X, Y)$.
- Consider an E-valued stochastic Volterra equation with càglàd paths:

$$Z_t = Z_0 + \int_0^t K(t-s)b(s)ds - \int_0^t K(t-s)BZ_s ds + \int_0^t K(t-s)dM_s,$$

where

- $Z_0 \in E$
- K denotes a matrix valued kernel in $L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ of block diagonal form $K = \begin{pmatrix} K_X & 0 \\ 0 & K_Y \end{pmatrix}$.
- M denotes an $\mathbb{R}^d$-valued martingale such that each component is in $\mathcal{H}^2$ and $\langle M, M\rangle_{t,ij} = \int_0^t c_{ij}(Z_s)ds + \int_0^t \int_{\mathbb{R}^d} \xi_i \xi_j F(Z_s, d\xi)ds$ for some function $c : \mathbb{R}^d \to \mathbb{S}^d_+$ and some Borel kernel F from $\mathbb{R}^d$ into $\mathbb{R}^d$.
- $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^d)$
- $B \in \mathbb{R}^{d\times d}$

Resolvents - Notation

- The resolvent of K is the kernel $R \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ that satisfies

$$K * R = R * K = K - R.$$

- We shall also consider the resolvent of the first kind whenever it exists. This is an $\mathbb{R}^{d\times d}$-valued measure on $\mathbb{R}_+$ of locally bounded variation such that

$$K * L = L * K = I_d.$$

- We write R_B for the resolvent of KB and $N_B := K - R_B * K$.

Examples

- $K(t) = 1, \quad R(t) = \exp(-t), \quad L(dt) = \delta_0(dt);$
- $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad R(t) = t^{\alpha-1} E_{\alpha,\alpha}(t^{-\alpha}), \quad L(dt) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} dt$

"Variance swap curves" and their state space

- Our first goal: Find a Markovian structure behind these Volterra equations
- Let Z be a solution of the above stochastic Volterra equation such that it has finite first moment for every $t \geq 0$. Define the "variance swap curve" process

$$W_t(x) := \mathbb{E}[Z_{t+x} | \mathcal{F}_t]$$

and note that $W_t(0) = Z_t$.

Proposition (C., Teichmann (2017))

Assume that K admits a resolvent L of the first kind such that it is non-increasing in the following sense that $L_X([s, s+t])x \preceq L_X([0, t])x$. Then the function valued process $(x \mapsto W_t(x))_{t \geq 0}$ takes values in

$$\mathcal{E} = \Big\{ f : \mathbb{R}_+ \to E \,|\, f(x) = (I_d - \int_0^x R_B(s) ds) V + \int_0^x N_B(x-s) h(s) ds$$
$$\text{with } V \in E,\ h_X(x) dx \succeq -L_X(dx) V_X \Big\}.$$

"Variance Swap curves" and associated SPDE

Proposition (C., Teichmann (2017))

The variance swap curve process $(W_t)_{t\geq 0}$ is a time-homogenous Markov process on $\mathcal{E}$ and satisfies the SPDE

$$dW_t(x) = \frac{d}{dx} W_t(x) dt + N_B(x) dM_t,$$
$$W_0(x) = \left(I_d - \int_0^x R_B(s) ds \right) Z_0 + \int_0^x N_B(x-s) b(s) ds,$$

in the following mild pointwise ("Walsh") sense

$$W_t(x) = S_t W_0(x) + \int_0^t S_{t-s} N_B(x) dM_s$$
$$= W_0(x+t) + \int_0^t N_B(x+t-s) dM_s,$$

where $(S_t)_{t\geq 0}$ denotes the shift semigroup.

Remarks

- Embedding the state space $\mathcal{E}$ in an appropriate Hilbert space allows to consider solutions of the above SPDE in the usual mild sense.
- Existence of stochastic Volterra equations for enough curves b can be translated to existence of the above SPDE and vice versa.
- Note that by construction via $\mathbb{E}[Z_{t+x}|\mathcal{F}_t]$, $W_{X,t}(x) \in C$ for all $x \geq 0$. This necessarily implies that whenever $W_{X,t}(x) \in \partial C$ for some $x > 0$ that $W_{X,t}(0) \in \partial C$ because this enters in the volatility. This is one of the invariance properties the state space $\mathcal{E}$ satisfies.

Affine characteristics - Setting

- For notational simplicity we let $E = C$.
- We now take the martingale of the following form

$$M_t = \int_0^t \sqrt{c(W_t(0))} dB_t + \int_0^t \int_{\mathbb{R}^d} \xi(\mu^M(d\xi, dt) - F(W_t(0), d\xi)dt,$$

  where B is a d-dimensional Brownian motion, μ^M the random measure of the jumps and F its compensator.
- We assume that the characteristics c and F are linear, i.e.

$$c(x) = \sum_{i=1}^d c_i x_i, \quad c_i \in \mathbb{S}^d \text{ s.t. } \quad c(x) \in \mathbb{S}_+^d \text{ on } E,$$
$$F(x, d\xi) = x^\top \nu(d\xi)$$

  where ν is a d-dimensional vector valued measure on $\mathbb{R}^d$ with bounded support s.t. $F(x, d\xi)$ is a measure for every $x \in E$.

Affine characteristics - Setting

- Define the polar cone of $\mathcal{E}$

$$\mathcal{U} = \{\mu \in \mathcal{M} \,|\, \langle f, \mu \rangle \leq 0 \quad \forall f \in \mathcal{E}\},$$

  where $\mathcal{M}$ denotes the set of d-dimensional vector valued signed measures on $\mathbb{R}_+$. For a function $f \in \mathcal{E}$ and $\mu \in \mathcal{M}$ we write $\langle f, \mu \rangle = \int_0^\infty f^\top(x)\mu(dx)$.

- For a function $V : \mathcal{M} \to \mathbb{R}$, $\partial_\mu V$ denotes the directional derivative in directions δ_x, i.e.

$$\lim_{\varepsilon \to 0} \frac{V(\mu + \varepsilon \delta_x) - V(\mu)}{\varepsilon}$$

Existence assumption

Assumption A

Assume that the SPDE

$$dW_t(x) = \frac{d}{dx} W_t(x)dt + N_B(x)dM_s, \quad W_0 = f \in \mathcal{E},$$

with M an affine martingale admits a weak solution with values in $\mathcal{E} \subseteq C(\mathbb{R}_+, E)$, i.e. for every $\mu \in \mathcal{M} \cap \mathcal{D}(-\frac{d}{dx})$

$$\begin{aligned}\int W_t^\top(x)\mu(dx) &= \int W_0^\top(x)\mu(dx) - \int_0^t \int W_s^\top(x)\frac{d}{dx}\mu(dx)ds \\ &+ \int_0^t \left(\int \sqrt{c(W_t(0))}^\top N_B(x)^\top \mu(dx)\right)^\top dB_s \\ &+ \int_0^t \left(\int N_B(x)^\top \mu(dx)\right)^\top \int_{\mathbb{R}^d} \xi(\mu(d\xi, dt) - F(W_t(0), d\xi)dt).\end{aligned}$$

Affine transform formula

Theorem (C., Teichmann (2017))

Under assumption A, $(W_t(x))_{t\geq 0}$ is an affine process on $\mathcal{E}$ in the sense that for all $\mu \in \mathcal{U} \cap \mathcal{D}(-\frac{d}{dx})$

$$\mathbb{E}_w\left[\exp(\langle W_t, \mu\rangle)\right]$$

solves the following transport equation

$$\partial_t V(t,\mu,w) = \langle R(\mu), \partial_\mu V(t,\mu,w)\rangle, \quad V(0,\mu,w) = \exp(\langle \mu, w\rangle),$$

where $R(\mu)$ is given by

$$\begin{aligned} R(\mu)(dx) &= -\frac{d}{dx}\mu(dx) + \frac{1}{2}\delta_0(dx) C\Big(\int N_B^\top(x)\mu(dx)\Big) \\ &\quad + \delta_0(dx)\int \left(e^{(\int N_B^\top(x)\mu(dx))^\top \xi} - 1 - \Big(\int N_B^\top(x)\mu(dx)\Big)^\top \xi\right)\nu(d\xi) \end{aligned}$$

with $C(u) := (u^\top c_1 u, \ldots, u^\top c_d u)^\top$.

Affine transform formula

Theorem (cont.)

Assume that we have a mild solution to the Riccati PDE, i.e.

$$\psi(t,\mu)(dx) = \mu(dx-t) + \frac{1}{2}\int_0^t \delta_0(dx-t+s)C(\widetilde{\psi}(s,\mu))ds$$
$$+\int_0^t \delta_0(dx-t+s)\int \left(e^{(\widetilde{\psi}(s,\mu))^\top \xi} - 1 - (\widetilde{\psi}(s,\mu))^\top \xi\right)\nu(d\xi)ds$$

where $\widetilde{\psi}(t,\mu) = \int N_B^\top(x)\psi(t,\mu)(dx)$. *Then the unique solution of the transport equation is given by* $\exp(\langle \psi(t,\mu), w\rangle)$ *so that*

$$\mathbb{E}_w\left[\exp(\langle W_t,\mu\rangle)\right] = \exp(\langle \psi(t,\mu), w\rangle).$$

Connection to Riccati Volterra equations

- Approximate $\mu(dx) = u\delta_0(dx)$ with $u \in C^*$.
- Define $\widetilde{\psi}(t) := \int_0^\infty N_B^\top(x)\psi(t, u\delta_0)(dx)$.
- Then the above result translates to

$$\mathbb{E}_w\left[\exp(W_t(0)^\top u)\right] = \exp\left(w(t)^\top u + \frac{1}{2}\int_0^t w(s)^\top C(\widetilde{\psi}(t))ds\right)$$
$$\times \exp\left(\int_0^t w(s)^\top \int \left(e^{\widetilde{\psi}^\top(t)\xi} - 1 - \widetilde{\psi}^\top(t)\xi\right)\nu(d\xi)ds\right).$$

- Since $W_t(0)$ satisfies the stochastic affine Volterra equation

$$W_t(0) = V + \int_0^t K(t-s)h(s)ds - \int_0^t K(t-s)BW_s(0) + \int K(t-s)dM_s$$

this give the (Fourier)-Laplace transform of affine Volterra processes in terms of their variance swaps.

Connection to Riccati Volterra equations

- In the diffusion case this is the same representation as in Abi Jaber, Larsson, Pulido (2017), since

$$\widetilde{\psi}(t) := \int_0^\infty N_B^\top(x)\psi(t, \delta_0 u)(dx)$$

satisfies the generalized Riccati Volterra equation

$$\widetilde{\psi}(t) = N_B^\top(t)u + \int_0^t N_B^\top(t-s)C(\widetilde{\psi}(s,u))ds \\ + \int_0^t N_B^\top(t-s)\left(\int \left(e^{\widetilde{\psi}^\top(s)\xi} - 1 - \widetilde{\psi}^\top(s)\xi\right)\nu(d\xi)\right)ds$$

Towards existence & uniqueness - via standard affine processes

- Consider a standard affine process with time-inhomogeneous "constant" drift $t \mapsto b(t)$ on $E = \mathbb{R}_+^m \times D$ with $D \subseteq \mathbb{R}^n$ of the form $Z = (X, Y)$

$$Z_t = Z_0 + \int_0^t b(s)ds + \int_0^t AZ_s ds + \int_0^t \underbrace{\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}}_{(m+n)\times(m+n)} dB_s,$$

  where B is an $m+n$-dimensional Brownian motion $\Sigma = \mathrm{diag}(\sigma_1\sqrt{X_{t,1}}, \ldots, \sigma_m\sqrt{X_{t,m}})$ and A an $\mathbb{R}^{(n+m)\times(n+m)}$ matrix.

- By variation of constants the $\mathbb{R}_m^+$-valued process

$$X_t = (e^{At}Z_0)_X + \int_0^t (e^{A(t-s)}b(s))_X ds + \int_0^t (e^{A(t-s)})_{XX} \Sigma dB_{s,X},$$

  is of Volterra form.

Towards existence & uniqueness - via standard affine processes

- The assumptions in the above theorem on the primal side (existence of $W_t(x) = \mathbb{E}[X_{t+x}|\mathcal{F}_t]$) and on the dual side (Riccati equations) are satisfied. In particular we have the following representation of the Riccati PDE in terms of the characteristic exponent $\widehat{\psi}$ of the original affine process Z

$$\begin{aligned}
&\psi(t,\mu)(dx) \\
&= \mu(dx-t) + \frac{1}{2}\int_0^t \delta_0(dx-t+s) C\Big(\int (e^{Ax})_{XX}\psi(t,\mu)(dx)\Big) \\
&= \mu(dx-t) + \frac{1}{2}\int_0^t \delta_0(dx-t+s) C\Big(\widehat{\psi}_X\big(t, \int (e^{A^\top x}(\mu(dx),0)^\top)\big)\Big)
\end{aligned}$$

Towards existence & uniqueness - kernel approximation

- Consider a sequence of matrices $A^n \in \mathbb{R}^{(m+n)\times(m+n)}$ growing in the dimension and σ_i^n for $i \in \{1, \ldots, m\}$. Then we can consider the following types of kernels

$$K(x) = \lim_{n\to\infty} (e^{A^n x})_{XX} \operatorname{diag}(\sigma_1^n, \ldots, \sigma_m^n) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{m\times m}).$$

- We can go beyond diagonal kernels. In particular, in the one-dimensional case, completely monotone kernels can be generated.
- For these types of kernels, we (conjecture to) obtain
  - (probabilistically) strong (PDE) weak solutions of the variance swap curve SPDE $(W_t)_{t\geq 0}$;
  - existence of global solutions to the Riccati equations;
  - uniqueness in law;
  - numerical approximations by standard affine processes.

Conclusion

- We provide a generic Markovian structure for general Volterra equations via the variance swap curve process $(W_t)_{t \geq 0}$.
- In the case of affine characteristics, we have an affine transform formula for the process $(W_t)_{t \geq 0}$.
- For a big class of kernels obtained as limit of scaled entries of matrix exponentials, we obtain numerical approximations in the case of affine characteristics.

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Thank you for your attention!