# Robust Hedging of Options on a Leveraged Exchange Traded Fund

(Discretisation and Duality of Optimal Skorokhod Embedding Problems)

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Advances in Stochastic Analysis for Risk Modeling
CIRM, Luminy, 17th November, 2017

# Model-independent/Robust bounds for option prices

- Aim: make statements about the price of options given very mild modelling assumptions
- Incorporate market information by supposing the prices of vanilla call options are known
- Typically want to know the largest/smallest price of an exotic option (Lookback option, Barrier option, Variance option, Asian option,. . . ) given observed call prices, but with (essentially) no other assumptions on behaviour of underlying
- This talk: options on *Leveraged Exchange Traded Funds (LETF)*
- Why? Heavily traded, and interesting features to the solution!

# Leveraged Exchange Traded Fund (LETF)

- ETF attempts to match returns on a benchmark asset/index 1:1
- LETF attempts to match returns on a benchmark asset/index up to factor, e.g. 2:1 — 10% increase in index → 20% increase in LETF
- Over time, e.g. daily rebalancing leads to tracking errors
- Dynamics of the LETF with leverage ratio $\beta > 1$ are given by

$$L_t = S_t^{\beta} \exp\left( -\frac{\beta(\beta - 1)}{2} V_t \right),$$

  $V_t$ is the accumulated quadratic variation of $\log S_t$

- Eliminate $V_t$ by time change, $\tau_t := \inf\{s \geq 0 : \ V_s = t\}$ and $X_t := S_{\tau_t}$. So,

$$\mathrm{d}\langle X \rangle_t = \mathrm{d}\langle S \rangle_{\tau_t} = S_{\tau_t}^2 \mathrm{d}V_{\tau_t} = X_t^2 \mathrm{d}t$$

  and $X_t$ is a geometric Brownian motion (GBM)

# LETF model-independent pricing problem

Want to consider (maximum) price of call option on LETF under assumption that law of $S_T$ (under $\mathbb{Q}$) is known, but no other modelling assumption. Corresponds to:

**Main Problem**

Find

$$\sup_{\tau} \mathbb{E}\left[\left(X_\tau^\beta \exp\left(-\frac{\beta(\beta-1)}{2}\tau\right) - k\right)_+\right], \qquad \text{(LOptSEP)}$$

over stopping times $\tau$ such that $X_\tau \sim \mu$, where $X$ is a GBM

- This is a form of Optimal Skorokhod Embedding Problem (OptSEP): Given $\gamma$ an optional process and $\mu$ probability on $\mathbb{R}$, find stopping time $\tau$ to maximise

$$\mathbb{E}[\gamma_\tau] \qquad \text{(OptSEP)}$$

subject to $B_\tau \sim \mu$, $(B_{t\wedge\tau})_{t\geq 0}$ is UI, $B$ is Brownian motion.

# Monotonicity principle

- In Beiglböck, C., Huesmann [BCH] (2017), systematic study of all known optimal solutions
- Key observation of [BCH]:

  *Solutions to (OptSEP) are often characterised by simple geometric criteria*

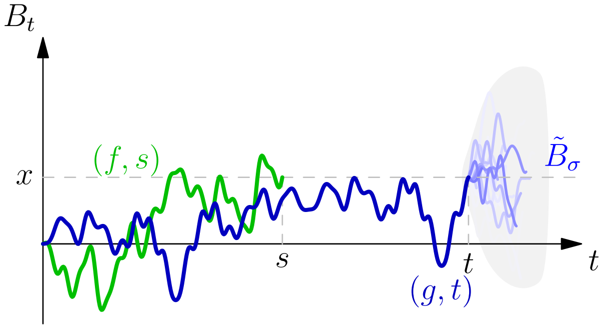
- Geometric criteria typically determined by the monotonicity principle ([BCH]):

  *if I am better off 'stopping' a currently running path, and 'transplanting' the tail onto another stopped path (stopping at the same level), my solution is not optimal*

- Monotonicity principle can be used to show that optimisers of (OptSEP) have a certain geometric form

# Monotonicity Principle

Stopped paths: $\mathsf{S} := \{(f, s) | s \in \mathbb{R}_+, f \in C[0, s]\}$. Objective, $\gamma : S \to \mathbb{R}$.




"Reward": $\gamma((f, s)) + \mathbb{E}\left[\gamma((g, t) \oplus \tilde{B}_\sigma)\right]$

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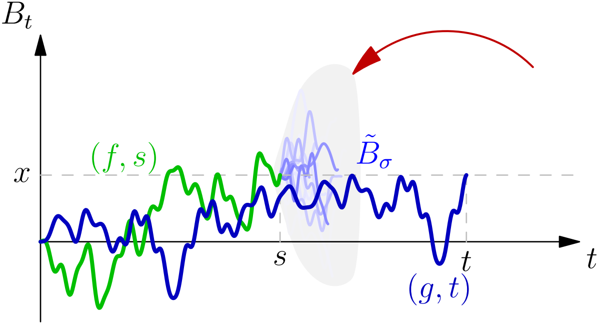


"Reward": $\mathbb{E}\left[\gamma((f,s) \oplus \tilde{B}_\sigma)\right] + \gamma((g,t))$

Stop-Go Pair: $((f,s),(g,t)) \in \mathsf{SG}^\gamma \iff f(s) = g(t)$ & $\forall$ st. t. $\sigma > 0$:

$$\gamma((f,s)) + \mathbb{E}\left[\gamma((g,t) \oplus \tilde{B}_\sigma)\right] > \mathbb{E}\left[\gamma((f,s) \oplus \tilde{B}_\sigma)\right] + \gamma((g,t))$$

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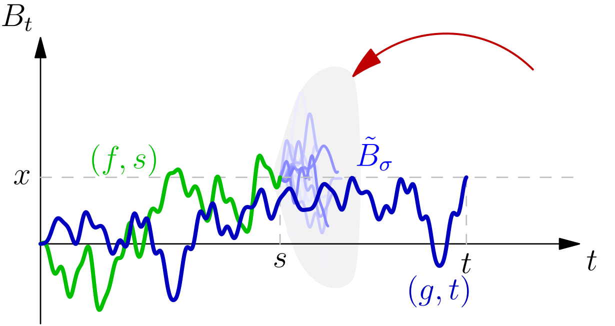


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## Theorem ([BCH] (2017))

*Let* $\tau^*$ *be an optimiser to (OptSEP). Then there exists* $\Gamma \subset \mathsf{S}$ *such that*

$$\mathbb{P}(((B_t)_{t\leq\tau},\tau)\in\Gamma)=1$$

*and*

$$\left(\Gamma^{<}\times\Gamma\right)\cap \mathsf{SG}^{\gamma}=\emptyset,$$

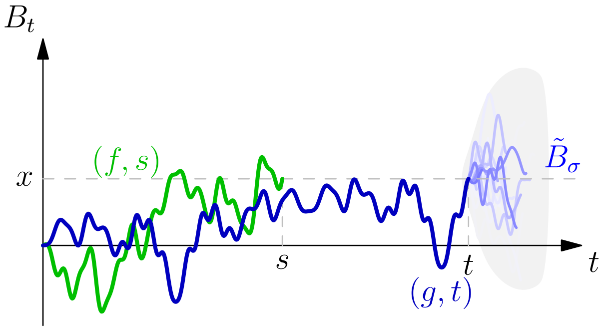
*where*

$$\Gamma^{<}:=\{(g,t):\exists(f,s)\in\Gamma, s>t, f|_{[0,t]}=g\}.$$

- Sufficient to recover every known optimal solution (at the time of writing)

# Example: Rost's embedding

Consider $\gamma_t = F(t)$, $F$ a convex function:



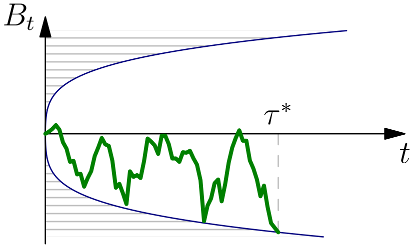

Score either: $F(s) + \mathbb{E}[F(t+\sigma)]$ or $\mathbb{E}[F(s+\sigma)] + F(t)$.
If $s < t$, we see that $F(s) + \mathbb{E}[F(t+\sigma)] > \mathbb{E}[F(s+\sigma)] + F(t)$ and so $((f,s),(g,t)) \in \mathsf{SG}^\gamma$.

# Example: Rost's embedding

- Monotonicity Principle implies stopping region is a reversed barrier:

$$\tau^* = \inf\{t \geq 0 : (t, B_t) \in \mathcal{R}\}$$

where $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$, $(s, x) \in \mathcal{R} \implies (t, x) \in \mathcal{R}$ for all $t < s$




- Loynes' argument says that there is (essentially) one such barrier (take union of two barriers, also embeds same law)

# Example: LETF model-independent pricing problem

- Problem is to maximise $\mathbb{E}[(M_\tau - k)_+]$, where $M_t = X_t^\beta \mathrm{e}^{-\beta(\beta-1)t/2}$ is a martingale. Intuitively, aim to maximise local time of $M$ at $k$
- Can compute $M_t = k$ when $K(X_t) = t$, $K(x) = \frac{2}{\beta(\beta-1)} \ln(\frac{x^\beta}{k})$
- A *K-cave barrier* is a subset $\mathcal{R}$ of $\mathbb{R}_+ \times \mathbb{R}_+$ of the form $\mathcal{R} = \{(t,x) : t \leq \ell(x) \text{ or } t \geq r(x)\}$, where $\ell(x) \leq K(x) \leq r(x)$

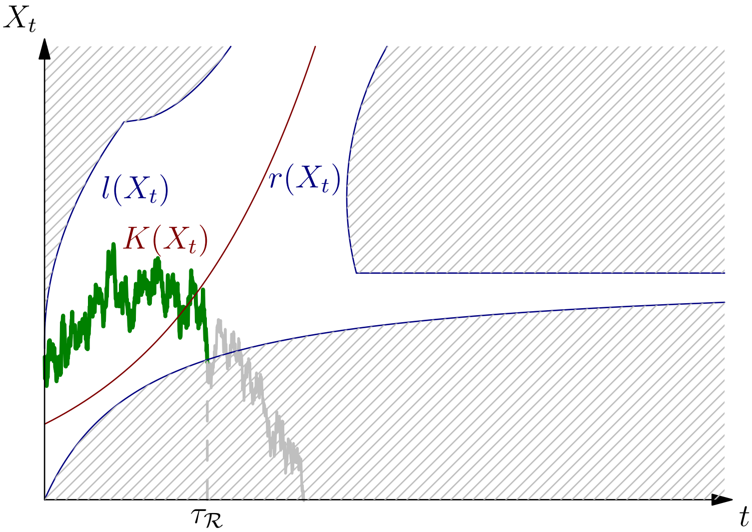
**Theorem**

*There exists an optimiser to (LOptSEP) which is of the form*

$$\tau_{\mathcal{R}} := \inf\{t > 0 : (t, X_t) \in \mathcal{R}\}$$

*where $\mathcal{R}$ is a $K$-cave barrier.*

# $K$-cave barriers



# (Non-)uniqueness of cave-type Barriers

- *Normally*, at this point, Loynes' argument would imply that there is a unique ($K$-)cave barrier with the right stopping distribution, which would then be the optimiser.
- However, for the ($K$-)cave barriers, there are generally multiple ($K$-)cave barriers which embed the same distribution; consider 3-atom measures. Crucial question:

  *How to identify the optimal $K$-cave barrier?*

# PDE Heuristics for the Dual Solution (LETF case)

- We expect the Dual solution (superhedging portfolio) to take the form: $\exists G, \lambda$ such that

$$G(t,x) + \lambda(x) \geq F(t,x),$$

where $\lambda$ represents a portfolio of calls, $F$ is the payoff of the option, and $\gamma$ is the proceeds of a dynamic trading strategy in the underlying.

- We argue heuristically, inspired by arguments of Henry-Labordère: write $F^\lambda(t,x) = F(t,x) - \lambda(x)$. Then we require:

$$\mathcal{L}G := \frac{x^2}{2}\partial_x^2 G + \partial_t G \leq 0 \text{ and } G \geq F^\lambda \qquad \forall (t,x)$$

and expect equality in PDE in $\mathcal{R}^\complement$, and $G = F^\lambda$ in $\mathcal{R}$.

- Also conjecture smooth fit: $\partial_t G = \partial_t F^\lambda = \partial_t F$ on boundaries

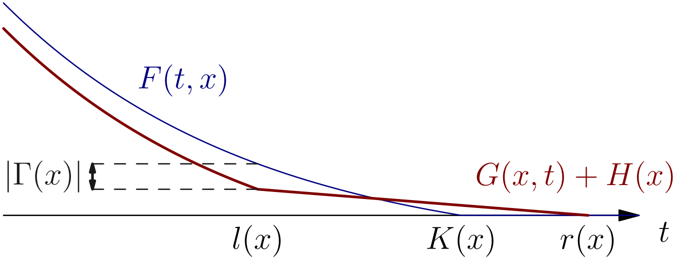
$$\implies M := \partial_t G \text{ solves } \mathcal{L}M = 0 \text{ in } \mathcal{R}^\complement \text{ and } M = \partial_t F \text{ on } \partial\mathcal{R}$$

# PDE Heuristics for the Dual Solution

- In particular, we get: $M(t,x) = \mathbb{E}^{(t,x)}[\partial_t F(X_{\tau_{\mathcal{R}}}, \tau_{\mathcal{R}})]$, and integrating, we see that
$$G(t,x) = \int_t^{r(x)} M(s,x)\,\mathrm{d}s - Z(x)$$
for some function $Z$.
- In fact, $Z$ can be chosen (uniquely up to affine functions) in such a way to make $G$ a martingale in $\mathcal{R}^{\complement}$.
- Now $G(t,x) \geq F^{\lambda}(t,x)$ at $t = \ell(x), t = r(x)$ implies that:

$$\lambda(x) \geq Z(x) + \max\{\underbrace{\quad 0 \quad}_{t=r(x)}, \underbrace{F(\ell(x),x) - \int_{\ell(x)}^{r(x)} M(s,x)\,\mathrm{d}s}_{\substack{:=\Gamma(x) \\ t=\ell(x)}}\}$$

# Γ-condition: $\Gamma > 0$




**Lemma (Easy)**

Suppose $\mathcal{R}$ is a $K$-cave barrier which embeds $\mu$, and such that $\Gamma(x) = 0$ for all $x$. Then $\tau_{\mathcal{R}}$ is an optimiser of (LOptSEP).

**Theorem (Hard)**

There exists a (unique) $K$-cave barrier $\mathcal{R}$ which embeds $\mu$, and such that $\Gamma(x) = 0$ for all $x$.

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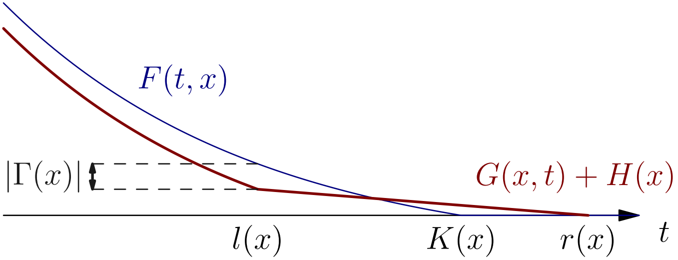


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# Dual Feasibility

First step to proving the results:

**Theorem**

*The dual solution described above is indeed a dual solution (i.e. $G$ is a martingale for some suitable $Z$).*

- Shown using essentially probabilistic techniques
- NB: No 'explicit' form for $Z$
- Clearly $\Gamma = 0$ is then a sufficient condition $\implies$ primal = dual
- In fact, enough to show

$$\Gamma(x) \leq 0 \quad \mu_r\text{-a.s.}, \qquad \Gamma(x) \geq 0 \quad \mu_l\text{-a.s.}$$

  where $\mu_l, \mu_r$ are left and right exit distributions of optimiser.
- But: Not enough for theorem... Know (e.g. Dolinsky & Soner) that no duality gap, but don't know optimal dual solution of this form

- Idea: Take the original problem, and discretise time and space suitably (Random walk converging to the original Brownian motion)
- Can formulate the original problem in the discrete setting and formulate (LOptSEP) as a (countably infinite) linear programming problem
- Strong duality holds (in an appropriate sense) for the discretised problem, and can show existence of dual solutions, and natural condition corresponding to $\Gamma = 0$
- In the limit, exists optimal barrier, and embedding, and can make sense of $\Gamma = 0$ condition

$\implies$ *Theorem holds*

# Discretisation

- Fix a (centred, integrable) measure $\mu$, supported on $[x_*, x^*]$. Fix $N$.
- Define $\mathcal{J} := \{\lfloor x_* \sqrt{N} \rfloor, \lfloor x_* \sqrt{N} \rfloor + 1, \ldots, \lfloor x^* \sqrt{N} \rfloor\}$, and $x_j^N = \frac{j}{\sqrt{N}}$.
- Can construct a discrete time random walk $Y_j^N$ from hitting times of Brownian motion $B$ to $\{x_0^N, \ldots, x_L^N\}$, $L = |\mathcal{J}| - 1$.
- By considering a stopping rule $\tau$ for $B$, can construct stopping rule $\tilde{\tau}^N$ for $Y$. Then consider probabilities:

$$p_{j,n}^N := \mathbb{P}\left(Y_n^N = x_j^N, \, \tilde{\tau}^N \geq n+1\right)$$
$$q_{j,n}^N := \mathbb{P}\left(Y_n^N = x_j^N, \, \tilde{\tau}^N = n\right).$$

- Recall, wanted to optimise $F(t, B_t)$. Write $t(n) = n/N$ for 'time' corresponding to step $n$ of $Y$. Then $\bar{F}_{j,n} := F(t(n), x_j)$.

# First LP problem

Then the optimisation problem for the discrete process can formally be rewritten as:

$$\mathcal{P}^{N'} : \text{ maximise} \sum_{\substack{j\in\mathcal{J}\\ n\geq 1}} \bar{F}^N_{j,n} q_{j,n} \quad \text{over } (p_{j,n})_{\substack{j\in\mathcal{J}'\\ n\geq 1}}, (q_{j,n})_{\substack{j\in\mathcal{J}\\ n\geq 1}}$$

subject to:

- $p_{j,n}, q_{j,n} \geq 0, \quad \forall j, n$ • $\sum_{n=1}^{\infty} q_{j,n} = \mu^N(\{x^N_j\}), \quad \forall j \in \mathcal{J}$
- $p_{j,n} + q_{j,n} = \frac{1}{2}(p_{j-1,n-1} + p_{j+1,n-1}), \quad \forall n \geq 2, j \in \mathcal{J}''$
- $p_{j^N_1,n} + q_{j^N_1,n} = \frac{1}{2} p_{j^N_2,n-1}, \quad p_{j^N_{L-1},n} + q_{j^N_{L-1},n} = \frac{1}{2} p_{j^N_{L-2},n-1}, \quad \forall n \geq 2$
- $p_{j^*+1,1} + q_{j^*+1,1} = \frac{1}{2},$ • $p_{j^*-1,1} + q_{j^*-1,1} = \frac{1}{2}$
- $p_{j,1} + q_{j,1} = 0, \quad \forall j \neq j^*, j^N_0, j^N_L,$ • $q_{j^N_0,1} = 0 = q_{j^N_L,1}$
- $q_{j^N_L,n} = \frac{1}{2} p_{j^N_{L-1},n-1}, q_{j^N_0,n} = \frac{1}{2} p_{j^N_1,n-1}, \quad \forall n \geq 2$

# Discretisation

In principle, we could consider this problem, but direct approach doesn't seem to help... Introduce new formulation $\mathcal{P}^N(\lambda)$ where:

- Remove $q$ variable: embedding condition derived from potential:

$$\sum_{n=1}^{\infty} p_{j,n} \leq U_j \sim \int |x_j^N - y| \, \mu(\mathrm{d}y) - |x_j^N|$$

- Restrict to $p_{j,n}$ satisfying a decay requirement. Lemma: If $\pi_{j,n}$ is probability when only stopping at $x_*, x^*$, then $\pi_{j,n} \approx \rho^n m_j$, some $\rho \in (0,1)$.
- Fix $\lambda > \rho^{-1} > 1$. Then restrict to set

$$\ell^1(\lambda) := \left\{ (p_{j,n})_{j,n} : \sum_{j,n} |p_{j,n}| \lambda^n < \infty \right\}$$

# Dual problem

$$\mathcal{D}^N(\lambda) = \inf\left\{ \sum_{j\in\mathcal{J}'} \nu_j U_j + \frac{1}{2}\left(\eta_{j^*+1,1} + \eta_{j^*-1,1}\right) + \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \right\}$$

over $(\nu_j)_{j\in\mathcal{J}'}, (\eta_{j,t})_{\substack{j\in\mathcal{J},\\ t\geq 1}}$, where $(\nu,\eta) \in \ell^\infty(\lambda^{-1})$ and:

- $\eta_{j,t}, \nu_j \geq 0, \forall j, t$
- $\frac{1}{2}\left(\eta_{j+1,t+1} + \eta_{j-1,t+1}\right) - \eta_{j,t} - \nu_j$
$$\leq \bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right), \quad \forall j, t.$$

- Note: Close correspondence to previous (continuous) dual solution

# Main result

**Theorem**

*For $\mathcal{P}^N(\lambda), \mathcal{D}^N(\lambda)$, there is no duality gap, and the optimiser is attained in the dual problem.*

- Key point: can show 'core' of feasible solutions using properties of the probabilities $\pi_{j,n}$. Essentially, can 'run' stopped mass to the boundaries for 'interior' solutions to the primal problem.

# Primal attainment

**Lemma**

*Suppose $\mathcal{P}^N < \infty$. Then $\mathcal{P}^N$ is attained by some $p^* \in \ell^1$, and $\mathcal{P}^N = \mathcal{P}^N(\lambda)$.*

- Essentially, feasible region of $\mathcal{P}^N$ is compact in $\ell^1$.
- In general, do not expect $\mathcal{P}^N(\lambda)$ to be attained, but can 'cut-off' solutions in $\ell^1$ to make them feasible for $\ell^1(\lambda)$.
- Discrete version of Monotonicity Principle $\implies$ geometric form of optimiser.

# Dual optimisers

- Our motivation for looking at the dual solution is to understand the Γ condition
- In e.g. the LETF case, can also verify that the dual solution can be assumed to be in $\ell^\infty$ by a truncation argument
- This is sufficient to ensure that complementary slackness holds!
- In particular, we can show:

**Lemma**

*Condition* (Γ) *holds in the limiting $K$-cave barrier $\mathcal{B}^\infty$.*

And hence:

**Theorem**

*For a $K$-cave stopping time $\tau$ given by curves $l$, $r$, the condition* (Γ) *is necessary for optimality.*

# References

- "Robust Hedging of Options on a Leveraged Exchange Traded Fund", (With S. M. Kinsley). arXiv:1702.07169
- "Discretisation and Duality of Optimal Skorokhod Embedding Problems", (With S. M. Kinsley). arXiv:1702.07173

# Conclusions

- Considered solutions to SEP which could not be completely recovered from the Monotonicity Principle: geometric characterisation does not guarantee uniqueness
- Need an additional condition, based on dual solution to determine optimal stopping region
- Proof of sufficiency of condition based on discretisation and classical duality results
- Need subtle duality results to get dual existence

# Fenchel Duality

**Theorem (Borwein & Zhu, Theorem 4.4.3)**

*Let* $\mathsf{X}$ *and* $\mathsf{Y}$ *be Banach spaces, let* $f : \mathsf{X} \to \mathbb{R} \cup \{\infty\}$ *and* $g : \mathsf{Y} \to \mathbb{R} \cup \{\infty\}$ *be convex functions and let* $A : \mathsf{X} \to \mathsf{Y}$ *be a bounded linear map. Define the primal and dual values* $\mathsf{p}, \mathsf{d} \in [-\infty, \infty]$ *by the Fenchel problems*

$$\mathsf{p} = \inf_{x \in \mathsf{X}} \{f(x) + g(Ax)\}$$

$$\mathsf{d} = \sup_{y^* \in \mathsf{Y}^*} \{-f^*(A^* y^*) - g^*(-y^*)\}.$$

*Then* $\mathsf{p} = \mathsf{d}$*, and the supremum in the dual problem is achieved if either of the following hold*

1. $0 \in \mathrm{core}\,(\mathrm{dom}(g) - A\mathrm{dom}(f))$ *and* $f, g$ *are lower semi-continuous*
2. $A\mathrm{dom}(f) \cap \mathrm{cont}(g) \neq \varnothing$.

Hard part: show dual attainment.