Branching diffusion representation of semi-linear elliptic PDEs and numerical applications

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Advances in Stochastic Analysis for Risk Modeling

November 15, 2017

Motivations

- We aim to provide a new probabilistic representation for

$$\mathcal{L}u + f\big(u, Du\big) = 0 \ \text{ in } \mathcal{O}, \qquad u = h \ \text{ on } \partial\mathcal{O}$$

  where $\mathcal{O} \subset \mathbb{R}^d$ bounded domain, $\mathcal{L}$ infinitesimal generator of diffusion
- BSDE approach:
  - *Darling & Pardoux '97*, *Pardoux '99*, *Briand et al. '03*, . . .
  - Monotonicity assumption in y, *i.e.*, $y \mapsto f(x, y, z)$ is non-increasing
  - Linear/Quadratic growth in z
- Branching diffusion approach:
  - Generator f multivariate polynomial
  - No motonicity assumption in y
  - Polynomial growth in z
  - Numerical applications

Branching Diffusion and PDEs

- *Skorokhod '64*, given $\beta > 0$, $(p_\ell)_{\ell\in\mathbb{N}}$ p.m.f., a solution of

$$\partial_t u + \mathcal{L}u + \beta\left(\sum_{\ell\in\mathbb{N}} p_\ell u^\ell - u\right) = 0 \text{ in } [0,T)\times\mathbb{R}^d, \quad u(T,\cdot) = g \text{ in } \mathbb{R}^d$$

  writes as

$$u(x) = \mathbb{E}\left[\prod_{k=1}^{N_T} g(X_T^k)\right]$$

  where $(X_T^1, \ldots, X_T^{N_T})$ are the positions of particles alive at time T
- *Henry-Labordère et al. '14, '16,* extensions of this result for

$$\partial_t u + \mathcal{L}u + f(u, Du) = 0 \text{ in } [0,T)\times\mathbb{R}^d, \qquad u(T,\cdot) = g \text{ in } \mathbb{R}^d$$

  where f is a multivariate polynomial

Outline

Description

- Start from one particle at position $x \in \mathcal{O}$
  - Its position is given by

$$X_t^x = x + \int_0^t \mu(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s, \quad t \geq 0$$

  - Its time of death is given by $\tau \wedge \eta^x$ where $\tau \sim \mathrm{Exp}(\beta)$ and

$$\eta^x := \inf \left\{ t \geq 0;\ X_t^x \notin \mathcal{O} \right\}$$

  - If $\tau < \eta^x$, then it gives rise to $I \sim (p_\ell)_{\ell \in \mathbb{N}}$ particles
- Then each child particle follows the same but an independent dynamic as the mother particle
- Particles are indexed by a label $k \in \bigcup_{n \geq 0} \mathbb{N}^n$ and we denote by
  - X^k its position
  - T_k its time of death
  - I^k its number of children

Assumptions

- The branching diffusion is well-defined if
  - $\sum_{\ell \in \mathbb{N}} \ell p_\ell < \infty$
  - (μ, σ) are Lipschitz
- Assume that the branching diffusion goes extinct a.s.
  - Sufficient condition $\sum_{\ell \in \mathbb{N}} \ell p_\ell \leq 1$
  - Necessary and sufficient condition for branching Brownian motion

$$\beta \left(\sum_{\ell \in \mathbb{N}} \ell p_\ell - 1 \right) - \frac{\lambda_1}{2} \leq 0$$

where λ_1 is the first positive eigenvalue of the Laplacian operator in $\mathcal{O}$, see *Watanabe '64*

Outline

Main Result I

Consider the semi-linear PDE

$$\mathcal{L}u + \beta\left(f(u) - u\right) = 0 \ \text{ in } \mathcal{O}, \qquad u = h \ \text{ on } \partial\mathcal{O} \qquad (1)$$

where $\beta > 0$, $f(x,y) := \sum_{\ell\in\mathbb{N}} c_\ell(x)y^\ell$. Denote

$$\psi^x := \prod_{\substack{k \text{ such that} \\ X^k_{T_k} \notin \mathcal{O}}} h(X^k_{T_k}) \prod_{\substack{k \text{ such that} \\ X^k_{T_k} \in \mathcal{O}}} \frac{c_{I^k}(X^k_{T_k})}{p_{I^k}}$$

Theorem

Assume that

(i) $\eta^x < \infty$ *a.s.*

(ii) σ *is uniformly elliptic on* $\partial\mathcal{O}$

(iii) $\partial\mathcal{O}$ *satisfies an exterior cone condition*

(iv) $(\psi^x)_{x\in\mathcal{O}}$ *is uniformly integrable*

Then the map $u : x \mapsto \mathbb{E}[\psi^x] \in \mathcal{C}(\bar{\mathcal{O}})$ *is a viscosity solution of* (1)

Outline

Numerical Applications

- Estimation of solution to semi-linear elliptic PDE by using the Monte Carlo method
  - Alternative to BSDE and deterministic methods
  - Accuracy depends on the dimensionless CLT
- The main difficulty is to simulate the exit time and position of a diffusion from a domain
- We restrict to branching Brownian motion in a hyperrectangle and use the walk on square method for Brownian motion
  - Introduced by *Faure '92*, *Milstein & Tretyakov '99*,. . .
  - Implemented by *Lejay '09* in the library `exitbm`
  - Exact simulation

Examples

1D. Consider the following ODE

$$u'' + u^3 - u = 0 \quad \text{in } \mathcal{O} = (-0.3, 0.3)$$

with explicit solution $u(x) = \frac{\sqrt{2}}{\cosh(x)}$

x	Exact	Estimate	99% conf. int.	StdDev/Mean	Time (secs)
0	1.4142	1.4144	$[1.4134, 1.4153]$	0.2644	13
-0.2	1.3864	1.3859	$[1.3852, 1.3866]$	0.1872	26

Table: *Numerical results for* $\beta = 1$, $p_1 = \frac{1}{2}$, $p_3 = \frac{1}{2}$ *with* 10^6 *sample paths*

4D. Consider the following PDE

$$\Delta u - 8(u^3 - u) = 0 \quad \text{in } \mathcal{O} = (-0.3, 0.3)^4$$

with explicit solution $u(x) = \tanh(\sum_{i=1}^4 x_i)$

x	Exact	Estimate	99% conf. int.	StdDev/Mean	Time (secs)
$(0.1, 0, 0, 0)$	0.1003	0.1010	$[0.0996, 0.1024]$	5.2787	514
$(0.1, 0.1, 0.1, 0)$	0.2913	0.2903	$[0.2890, 0.2915]$	1.6685	818

Table: *Numerical results for* $\beta = \frac{5}{2}$, $p_1 = \frac{13}{21}$, $p_3 = \frac{8}{21}$ *with* 10^6 *sample paths*

Outline

Semi-linear PDE II

- Consider the semi-linear PDE

$$\mathcal{L}u + \beta\big(f(u, Du) - u\big) = 0 \text{ in } \mathcal{O}, \quad u = h \text{ on } \partial\mathcal{O} \qquad (2)$$

where $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$f(x, y, z) := \sum_{\ell=(\ell_0,\dots,\ell_m)\in\mathbb{N}^{m+1}} c_\ell(x) y^{\ell_0} \prod_{i=1}^{m} (b_i(x)\cdot z)^{\ell_i}$$

- Consider a larger class of branching diffusion
  - Age-dependent, *i.e.*, each particle lives a random time distributed according to p.d.f. $\rho(t) \neq \beta e^{-\beta t}$
  - Marked particles, *i.e.*, each particle gives rise to $|\ell| = \sum_{i=0}^{m} \ell_i$ particles with probability p_ℓ, among which ℓ_0 have mark 0, ℓ_1 have mark 1, ...

Main Result II

Denote

$$\psi^x := \prod_{\substack{k\in\mathcal{K}^x \\ X^k_{T_k}\notin\mathcal{O}}} \frac{e^{-\beta\Delta T_k} h(X^k_{T_k})}{\bar{F}(\Delta T_k)} \mathcal{W}_k \prod_{\substack{k\in\mathcal{K}^x \\ X^k_{T_k}\in\mathcal{O}}} \frac{\beta e^{-\beta\Delta T_k} c_{I^k}(X^k_{T_k})}{\rho(\Delta T_k) p_{I^k}} \mathcal{W}_k$$

where $\bar{F}(t) := \int_t^\infty \rho(s)\,ds$, $\Delta T_k := T_k - T_{k-}$, T_{k-} and m_k are the time of birth and the mark of particle k,

$$\mathcal{W}_k := \mathbf{1}_{m_k=0} + \mathbf{1}_{m_k\neq 0} b_{m_k}(X^k_{T_{k-}}) \cdot \mathcal{W}(\Delta T_k, X^k_{T_{k-}}, W^k)$$

Theorem

Assume that

(i) $\partial\mathcal{O}$ *is of class* $\mathcal{C}^2$

(ii) σ *is uniformly elliptic in* $\bar{\mathcal{O}}$

(iii) μ, σ, h *are of class* $\mathcal{C}^{1,\alpha}$ *in* $\bar{\mathcal{O}}$

(iv) $(\psi^x)_{x\in\mathcal{O}}$, $(\psi^x b_i(x) \cdot \mathcal{W}(\tau\wedge\eta^x, x, W))_{x\in\mathcal{O}}$ *are bounded in* L^1

Then the map $u : x \mapsto \mathbb{E}[\psi^x] \in \mathcal{C}^1(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$ *is a viscosity solution of* (2)

Automatic Differentiation Formula

Fix $T > 0$ and denote for $x \in \mathcal{O}$, $s \geq 0$,

$$\mathcal{W}(s, x, W) = \int_0^{\zeta_{s\wedge T}} \theta_{s\wedge T}^{-1}(r, X_r^x)\sigma^{-1}(X_r^x)Y_r^x \, dW_r$$

where Y^x is the tangent process and

$$\theta_s(r, y) := d\left(y, \partial\mathcal{O}\right)^2(s-r), \quad \zeta_s := \inf\left\{t > 0 \; : \; \int_0^t \theta_s^{-1}(r, X_r^x)\, dr = 1\right\}$$

Proposition (*Thalmaier '97, Delarue '03, Gobet '04*)

(i) *The map* $\varphi : x \mapsto \mathbb{E}[e^{-\beta\eta^x} h(X_{\eta^x}^x)] \in \mathcal{C}^1(\mathcal{O})$ *s.t.*

$$D\varphi(x) = \mathbb{E}\left[e^{-\beta\eta^x} h(X_{\eta^x}^x)\mathcal{W}(T, x, W)\right]$$

(ii) *For any* g *bounded, the map* $\psi : x \mapsto \mathbb{E}[\int_0^{\eta^x} e^{-\beta s} g(X_s^x)\, ds] \in \mathcal{C}^1(\mathcal{O})$ *s.t.*

$$D\psi(x) = \mathbb{E}\left[\int_0^{\eta^x} e^{-\beta s} g(X_s^x)\mathcal{W}(s, x, W)\, ds\right]$$

Thank you for your attention!