Volatility estimation for stochastic PDEs

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Advances in Stochastic Analysis for Risk Modeling

Two classical stochastic PDEs

a) Stochastic heat equation

$$(\partial_t - \partial_x^2)Y(t,x) = \dot{M}(t,x)$$
$$Y(0,x) \equiv 0$$

b) Stochastic wave equation

$$(\partial_t^2 - \partial_x^2)Y(t,x) = \dot{M}(t,x)$$
$$Y(0,x) \equiv \partial_t Y(0,x) \equiv 0$$

$$M(\mathrm{d}t, \mathrm{d}x) = \sigma(t,x)\, W(\mathrm{d}t, \mathrm{d}x)$$

1. σ predictable random field
2. W Gaussian space–time white noise

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Mild solution:

$$Y(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) \sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y)$$

where

$$\text{a) } G(t,x) = (4\pi t)^{-1/2} \mathrm{e}^{-\frac{x^2}{4t}} \mathbf{1}_{\{t>0\}}, \quad \text{b) } G(t,x) = \frac{1}{2} \mathbf{1}_{\{|x| \le t\}}$$

Stationarity assumption:

$$\boxed{Y(t,x)=\int_{-\infty}^{t}\int_{\mathbb{R}} G(t-s,x-y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)}$$

a) **Damped** heat equation: $G(t,x)=(4\pi t)^{-1/2}\mathrm{e}^{-\frac{x^2}{4t}-t}\mathbf{1}_{\{t>0\}}$

b) **Damped** wave equation: $G(t,x)=\frac{1}{2}\mathrm{e}^{-t}\mathbf{1}_{\{|x|\le t\}}$

Applications

- Vibrating string (Cabaña 70)
- Electrical potential of neurons (Tuckwell & Walsh 83)
- Astrophysics (Jones 99)
- Forward rates (Cont 05)
- Turbulence modeling (Barndorff-Nielsen, Benth & Veraart 11)
- Phytoplankton modeling (El Saadi & Benbaziz 15)
- Within mathematics (e.g., KPZ equation, Hairer 13)

Problem formulation

Given: Observations $Y(\Delta_n, x), Y(2\Delta_n, x), \ldots, Y([T/\Delta_n]\Delta_n, x)$ at **fixed *x*** where Δ_n is small

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Wanted: Information about σ^2 (**"volatility/intermittency"**)

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How: Study limit as $n \to \infty$ of **power variation**

$$V_p^n(Y,t) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \left| \frac{\Delta_i^n Y}{\tau_n} \right|^p, \quad t \in [0,T], \quad p > 0$$

where

- $\Delta_i^n Y = Y(i\Delta_n, x) - Y((i-1)\Delta_n, x)$
- τ_n appropriate scaling factor

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- $\Delta_i^n Y = Y(i\Delta_n, x) - Y((i-1)\Delta_n, x)$
- τ_n appropriate scaling factor, here:

$$\tau_n := \left(\iint_0^\infty (G(s - \Delta_n, y) - G(s, y))^2 \mathrm{d}s\mathrm{d}y \right)^{\frac{1}{2}} = \begin{cases} \Delta_n^{\frac{1}{4}} & \textbf{Heat eq.} \\ \Delta_n^{\frac{1}{2}} & \textbf{Wave eq.} \end{cases}$$

Related literature

Semimartingales: Jacod & Protter 12

Moving averages:

$$Y_t = \int_{-\infty}^{t} G(t-s)\sigma_s \,\mathrm{d}W_s$$

See Barndorff-Nielsen, Corcuera & Podolskij 11

Stochastic PDEs:

- Quadratic/quartic variation in the case $\sigma \equiv 1$:

  Swanson 07, Pospíšil & Tribe 07, Liu & Tudor 16

- σ deterministic: Bibinger & Trabs 17

Law of large numbers for $V_p^n(Y, t)$

Theorem (C., 2017, in preparation)

Assume that $\sigma(t, x)$ uniformly $L^{(2\vee p)+\epsilon}$-continuous on $[0, T] \times \mathbb{R}$.

- For the heat equation we have

$$V_p^n(Y, t) \xrightarrow{\text{ucp}} \mu_p \int_0^t |\sigma(s, x)|^p \, \mathrm{d}s$$

- For the wave equation

$$V_p^n(Y, t) \xrightarrow{\text{ucp}} \mu_p \int_0^t \left| \int_{\mathbb{R}} \frac{\sigma^2(s - |y|, x - y)}{\mathrm{e}^{2|y|}} \, \mathrm{d}y \right|^{\frac{p}{2}} \mathrm{d}s$$

where $\mu_p = \mathbb{E}[|X|^p]$ for $X \sim N(0, 1)$.

Why do we obtain different limits?

Limit behavior is determined by

$$\pi^n(A) := \frac{\iint_A (G(s-\Delta_n, y) - G(s,y))^2 \,\mathrm{d}s\,\mathrm{d}y}{\iint_0^\infty (G(s-\Delta_n, y) - G(s,y))^2 \,\mathrm{d}s\,\mathrm{d}y}, \quad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$$

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Heat equation:

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Heat equation:

$$\pi^n \xrightarrow{w} \delta_{(0,0)}$$

Wave equation:

$$\pi^n \xrightarrow{w} \pi, \quad \mathrm{supp}(\pi) = \{(|x|, x) \colon x \in \mathbb{R}\},$$

$$\bar{\pi}(B) = \pi(\{(|x|, x) \colon x \in B\}), \quad \bar{\pi}(\mathrm{d}x) = \mathrm{e}^{-2|x|}\,\mathrm{d}x$$

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Central Limit Theorem

Theorem (C., 2017, in preparation)

For the heat equation, if $p \geq 2$, under further assumptions on σ, we have

$$\Delta_n^{-\frac{1}{2}}(V_p^n(Y,t) - V_p(Y,t)) \stackrel{\text{st-}\mathcal{L}}{\Longrightarrow} Y$$

where Y lives on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ and, conditional on $\mathcal{F}$, is a centered Gaussian process with independent increments and variance

$$C_t = \left(v_p + 2 \sum_{r=1}^{\infty} \rho_p \left(\frac{1}{2}\sqrt{r+1} - \sqrt{r} + \frac{1}{2}\sqrt{r-1} \right) \right) \int_0^t |\sigma(s,x)|^{2p} \mathrm{d}s$$

where

$$v_p = \mathrm{Var}(|X|^p), \quad \rho_p(r) = \mathrm{Cov}(|X|^p, |Y|^p), \quad (X,Y) \sim N\left(0, \left(\begin{smallmatrix} 1 & r \\ r & 1 \end{smallmatrix}\right)\right)$$

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Feasible CLT $\Longrightarrow$ Confidence intervals!

Example

For $V_2^n(Y, t)$, the asymptotic variance is given by

$$C_t = \left(2 + 4\sum_{r=1}^{\infty}\left(\frac{1}{2}\sqrt{r+1} - \sqrt{r} + \frac{1}{2}\sqrt{r-1}\right)^2\right)\int_0^t \sigma^4(s, x)\mathrm{d}s$$
$$= 2.357487\int_0^t \sigma^4(s, x)\mathrm{d}s.$$

Thank you very much!