Obliquely Reflected Backward Stochastic Differential Equations

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Advances in Stochastic Analysis for Risk Modelling,
CIRM, 16 November 2017

Outline

Definition

- **Equation**: Let $\mathcal{D} \subset \mathbb{R}^d$ be an open convex domain: (Y, Z, K) solution to

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s + \int_t^T \mathrm{d}K_s$$

(C1) $Y_t \in \bar{\mathcal{D}}$ (constrained value process)

(C2) $\int_0^T \mathbf{1}_{\{Y_t \notin \partial \mathcal{D}\}} \mathrm{d}K_t = 0$ (''minimality'' condition)

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- **Direction of Reflection**? $\mathfrak{n}(y)$ denotes the (set of) outward normal for $y \in \partial\mathcal{D}$
  1. Normal reflection: $\mathrm{d}K_t = -\Phi_t \mathrm{d}t$, $\Phi_t \in \mathfrak{n}(Y_t)$.
  2. Oblique reflection: $\mathrm{d}K_t = -H(t, Y_t, Z_t)\Phi_t \mathrm{d}t$, H matrix valued s.t. $\nu(t, Y_t, Z_t) = H_t \mathfrak{n}(Y_t)$ is the oblique outward direction.

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- **Remarks**: - start from ν to build H symmetric $\hookrightarrow$ follows Lions & Sznitman (84)
  - we require $\langle \frac{\nu}{\|\nu\|}, \frac{\mathfrak{n}}{\|\mathfrak{n}\|} \rangle \geq \epsilon > 0$.
  - if $f = 0$, $Y_t \in \bar{\mathcal{D}}$ naturally $(K = 0)$.

Revisiting switching problems

- **(1) Classical switching problem**: e.g. Hamadene and Jeanblanc (01)
  - d modes of production, switching from one mode to another has a cost $c > 0$,
  - the strategy consists in: 1. a sequence of switching times (θ_j)
    2. the choice of production mode at each switch

  $\hookrightarrow$ the current state is a and the goal is to maximise the expected reward

  $$Y_0^1 = \max_a \mathbb{E}[J(a,0)] \text{ with } J(a,0) = \int_0^T f^{a_s}(s)\mathrm{d}s - \sum_{j\geq 0} c\mathbf{1}_{\{t\leq\theta_j\leq T\}}$$

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- **(2) Randomised switching problem**: the mode at each switch is chosen randomly according to some Markov chain (ξ_n) such that $\mathbb{P}(\xi_n = j|\xi_{n-1} = i) = p_{ij}$ and $p_{ii} = 0$. $\hookrightarrow$ The agent know the (p_{ij}). (Same optimisation)

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- **Solution**: A system of reflected BSDEs

  $$Y_t^i = \int_t^T f^i(s)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}W_s + \int_t^T \mathrm{d}K_s^i \;,\; i \in \{1,\dots,d\}\;,\; Y_t \in \bar{\mathcal{D}}$$

  for (1) $\mathcal{D} = \{\, y \in \mathbb{R}^d | y^i \geq \max_j (y_j - c)\;,\; 1 \leq i \leq d \,\}$ (+ minimality)

  for (2) $\mathcal{D} = \{\, y \in \mathbb{R}^d | y^i \geq \sum_j p_{ij} y_j - c\;,\; 1 \leq i \leq d \,\}$ (+ minimality)

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Known results

- Normal Reflection:
  1. Existence and uniqueness in classical L^2, Lipschitz setting for a fixed domain (Gegout-Petit & Pardoux 95),
     some extension to random domain (Ma & Yong 99)
  2. One dimensional case: Simply reflected BSDEs (El Karoui, Kapoudjian, Peng & Quenez 97), Doubly reflected BSDEs (Cvitanic & Karatzas) and many others (applications to math. finance)

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- Oblique Reflection:
  1. Reflection in an orthant: Ramasubramanian (02), $f := f(Y)$.
  2. Switching problem: $f^i(Y, Z) = f^i(Y, Z^i)$ (do not cover non-zero sum game), contributions by several authors: Hu & Tang (08), Hamadene & Zhang (09), C., Elie & Kharroubi (11)
  3. Attempt to general case: existence of a (weak) solution when $f = f(Y)$ and $H = H(t, Y)$ is Lipschitz in the Markovian setting. By Gassous, Rascanu & Rotenstein (15)

Penalised BSDEs

- Let $d(y,\mathcal{D})$ distance to convex, $\mathcal{P}(y)$ normal projection, $\mathfrak{n}(y)$ outward normal:

$$Y_t^m = \xi + \int_t^T f(Y_s^m, Z_s^m)\mathrm{d}s - \int_t^T Z_s^m \mathrm{d}W_s - \int_t^T H(s, Y_s^m, Z_s^m)\Phi_s^m \mathrm{d}s$$
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- Estimates yield $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ with norms uniform in m and

$$\sup_t \mathbb{E}\Big[md^2(Y_t^m, \mathcal{D})\Big] + \mathbb{E}\Big[\int_0^T |\Phi_s^m|^2 \mathrm{d}s\Big] \leq C \ .$$

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- Is there a limit and does it satisfy an oblique RBSDE?

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  1. Apply Ito's formula to $\|Y_t - Y'_t\|^2$... works well in the normal case, thanks to

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3. Apply Ito's formula to $\Gamma_t(\alpha, \beta)(Y_t - Y'_t)^\top H(t, Y_t)^{-1}(Y_t - Y'_t)$ where

$$\Gamma_t = e^{\alpha t + \beta \int_0^t \left(|\Phi_s| + |\Phi'_s| + |Z_s|^2\right) \mathrm{d}s}.$$

$\hookrightarrow$ Control of Γ uses BMO tools

Outline

Markovian case: Existence in very general setting for H (discontinuous OK) and any convex domain (+ some weak hypothesis on the law of the forward process)

Non markovian case: Existence *and* Uniqueness for smooth H and $\mathcal{D}$ + some conditions on the BMO integral $\mathbb{E}[\xi \mid \mathcal{F}_t]$.

Various applications to switching problem:
- RBSDEs for randomised switching is well posed,
- For $d = 2$ and RBSDEs for switching problem, *existence and uniqueness* for full dependence in Z for f.

Open questions: H only Lipschitz? Non Convex domain?