Orthogonal Decompositions in Hilbert A-Modules

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A-Modules

Let A be an Archimedean f-algebra with (multiplicative) unit e

Definition

An abelian group $(H, +)$ is an A-module if and only if an outer product $\cdot : A \times H \rightarrow H$ is well defined with the following properties, for each $a, b \in A$ and for each $x, y \in H$:

1. $a \cdot (x + y) = a \cdot x + a \cdot y$
2. $(a + b) \cdot x = a \cdot x + b \cdot x$
3. $a \cdot (b \cdot x) = (ab) \cdot x$
4. $e \cdot x = x$

Clearly, if $A = \mathbb{R}$, then we are defining a (real) vector space

Hilbert A-Modules

Definition

An A-module H is a pre-Hilbert A-module if and only if an inner product $\langle \, , \, \rangle_H : H \times H \longrightarrow A$ is well defined with the following properties, for each $a \in A$ and for each $x, y, z \in H$:

5. $\langle x, x \rangle_H \geq 0$, with equality if and only if $x = 0$
6. $\langle x, y \rangle_H = \langle y, x \rangle_H$
7. $\langle x + y, z \rangle_H = \langle x, z \rangle_H + \langle y, z \rangle_H$
8. $\langle a \cdot x, y \rangle_H = a \langle x, y \rangle_H$

Clearly, if $A = \mathbb{R}$, then we are defining a pre-Hilbert space.

Orthogonal complementation

- Let H be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module
- If $\varnothing \neq M \subseteq H$, then the *orthogonal complement* of M is the set

$$M^{\perp} = \{ y \in H : \langle x, y \rangle_H = 0 \quad \forall x \in M \}$$

- $\varnothing \neq M \subseteq H$ is a submodule if and only if for each $a, b \in A$ and $x, y \in M$

$$a \cdot x + b \cdot y \in M$$

- Given $\{x_i\}_{i=1}^n \subseteq H$ for some $n \in \mathbb{N}$, then the (module) span of $\{x_i\}_{i=1}^n$, denoted $\operatorname{span}_A \{x_i\}_{i=1}^n$, is the set

$$\left\{ x \in H : \exists \{a_i\}_{i=1}^n \subseteq A \text{ s.t. } x = \sum_{i=1}^n a_i \cdot x_i \right\}$$

Contributions

- The main contributions of this paper are:
  1. to provide conditions on A and H that will allow us to conclude that given a submodule M in a pre-Hilbert A-module H, it is complemented i.e.
  $$H = M \oplus M^{\perp}$$
  2. to conclude that $\mathrm{span}_A \{x_i\}_{i=1}^{n}$ is always complemented
  3. to provide some interesting (cool?) applications of the above result

Topological structure and standard result

- Define $N(x) = \langle x, x \rangle^{\frac{1}{2}}$ for all $x \in H$
- Convergence could be defined as

$$x_n \to x \overset{def}{\iff} N(x_n - x) \to 0$$

Definition

Let A be an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module. H is an Hilbert A-module if and only if H is d_H complete and

$$d_H(x, y) = d(N(x - y), 0).$$

Theorem

Let A be an f-algebra of $\mathcal{L}^0$ type and H a Hilbert A-module. If M is a submodule, then the following statements are equivalent:

(i) $H = M \oplus M^{\perp}$;

(ii) M *is* d_H *closed.*

Main application: (stochastic processes)

- Consider a discrete-time filtered space $\left\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, P\right\}$
- Denote $\mathbb{E}\left(\cdot||\mathcal{F}_t\right)$ by $\mathbb{E}_t\left(\cdot\right)$ for all $t \in \mathbb{N}_0$. We consider two spaces of processes $x = (x_t)_{t\in\mathbb{N}_0}$
    1. $x \in \mathcal{S}_0$ if and only if $x_0 = 0$ and x is adapted $\left(x_t \in \mathcal{L}^0\left(\mathcal{F}_t\right)\right.$ for all $\left.t \in \mathbb{N}_0\right)$
    2. $x \in M_0^{2,loc}$ if and only if $x \in \mathcal{S}_0$, $\mathbb{E}_{t-1}\left(x_t^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right)$, and $\mathbb{E}_{t-1}\left(x_t\right) = x_{t-1}$ for all $t \in \mathbb{N}$
- Given $x \in \mathcal{S}_0$, we define $\Delta_t x = x_t - x_{t-1}$ for all $t \in \mathbb{N}$
- $H = \left\{x \in \mathcal{S}_0 : \mathbb{E}_{t-1}\left(x_t^2\right) \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N}\right\}$
- $a \in A$ if and only if $a = (a_t)_{t\in\mathbb{N}}$ is predictable, that is, $a_t \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right)$ for all $t \in \mathbb{N}$

Main application: (continued)

- $+ : H \times H \to H$ to be such that $(x+y)_t = x_t + y_t$ for all $t \in \mathbb{N}_0$
- $\cdot : A \times H \to H$ to be such that

$$(a \cdot x)_0 = 0 \text{ and } (a \cdot x)_t = \sum_{s=1}^{t} a_s (x_s - x_{s-1}) = \sum_{s=1}^{t} a_s \Delta_s x \quad \forall t \in \mathbb{N}.$$

We can also define a generalized inner product $\langle \ , \ \rangle_H : H \times H \to A$ by $(x, y) \mapsto \langle x, y \rangle_H$ where the latter is the process

$$(\langle x, y \rangle_H)_t = \mathbb{E}_{t-1} ((\Delta_t x)(\Delta_t y)) \qquad \forall t \in \mathbb{N}$$

Stochastic processes

Theorem

$(H, +, \cdot, \langle\ ,\ \rangle_H)$ *is an Hilbert A-module.*

Theorem

$\left(M_0^{2,loc}, +, \cdot, \langle\ ,\ \rangle_H\right)$ *is an Hilbert A-module. In particular,* $M_0^{2,loc}$ *is a closed submodule of* H.

Martingale decompositions: Doob

- $H^{pre} = \left\{ x \in \mathcal{S}_0 : x_t \in \mathcal{L}^0\left(\mathcal{F}_{t-1}\right) \quad \forall t \in \mathbb{N} \right\}$

Corollary

$H^{pre} = \left(M_0^{2,loc}\right)^{\perp}$. In particular, for each $x \in H$ there exists a predictable process $x_{pre} \in H^{pre}$ and a conditionally square integrable martingale $x_{mar} \in M_0^{2,loc}$ such that $x = x_{pre} + x_{mar}$. Moreover, this decomposition is unique.

Martingale decompositions: Kunita-Watanabe

- Observe that, if x and y are two square integrable martingales, then they are orthogonal in our sense, that is $\langle x, y \rangle_H = 0$, if and only if they are strongly orthogonal

Corollary

Let $\{x_i\}_{i=1}^n \in M_0^{2,loc}$. For each $x \in M_0^{2,loc}$, there exist $\{a_i\}_{i=1}^n \subseteq A$ and $y \in M_0^{2,loc}$ such that

$$x = \sum_{i=1}^n a_i \cdot x_i + y \text{ and } \langle x_i, y \rangle_H = 0 \quad \forall i \in \{1, ..., n\}.$$

Moreover, this decomposition is unique, in the sense that y is uniquely determined.

Take home message

- We provide characterizations for complementability in Hilbert modules
- Applications:
  - Stochastic processes (Doob decomposition and Kunita-Watanabe decomposition)
  - Stricker's Lemma
  - Conditional version of Von Neumann-Koopman Decomposition Theorem