Mean Field Games of Controls

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Based on a joint work with C.-A. Lehalle (CFM - Imperial College)
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Mean Field Games (MFG) study **stochastic optimal control problems** with **infinitely many interacting controllers**.
Pioneering works of Lasry-Lions (2006) and Huang-Caines-Malhamé (2006)

Goal of the talk

Study MFGs where dynamics and costs depend on the distribution of controls.

Known as “Extended MFG" or “MFG of controls"
Gomes, Patrizi and Voskanyan (14, 16), Carmona and Lacker (14), Bensoussan and Graber (16), Carmona and Delarue (monograph, 17), Acciaio-Backhoff-Carmona (in preparation).

The equilibrium configuration in a MFG of controls

- m_0 = **initial population density** of small agents on the state space $\mathbb{R}^d$.
- Controlled SDE for a small agent with initial position $x_0 \in \mathbb{R}^d$:

$$\left\{ \begin{array}{l} dX_t = b(t, X_t, \alpha_t; \mu_t)dt + \sigma(t, X_t, \alpha_t; \mu_t)dW_t \\ X_0 = x_0 \end{array} \right.$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian Motion (idiosyncratic noise).

- **Anticipating the evolution of the population density** $(m_t)_{t\in[0,T]}$ **and of the (population$\times$control) density** $(\mu_t)_{t\in[0,T]}$, the agent solves the optimal control problem

$$\inf_{\alpha} \mathbb{E}\left[\int_0^T L(t, X_t, \alpha_t; \mu_t)\, dt + g(X_T, m_T)\right].$$

- Let $\alpha^* = \alpha_t^*(x)$ **be the optimal feedback** of the agent.
- Under a **mean field assumption**, the actual population density is given by $\tilde{m}_t = \mathcal{L}(X_t^*)$ where

$$\left\{ \begin{array}{l} dX_t^* = b(t, X_t^*, \alpha_t^*(X_t^*); \mu_t)dt + \sigma(t, X_t^*, \alpha_t^*(X_t^*); \mu_t)dW_t \\ X_0^* \equiv m_0 \end{array} \right.$$

- **Equilibrium configuration :** $\boxed{\mu = (id, \alpha^*)\sharp\tilde{m}.}$

Typical issues

- Existence/uniqueness
- Application to problems with finitely many agents
- Limit of the $N-$player problems
- The formation of the MFG equilibria (Learning)
- Potential MFG (variational aspects)

Outline

1. An example
2. MFG of controls : the degenerate case
3. The potential case

Outline

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An optimal liquidation problem with trade impact

Joint work with C.-A. Lehalle.
(Model inspired by Almgren-Chriss (00) - close to Carmona-Lacker (14))

- A continuum of investors indexed by $a \in A$ decide to buy or sell a given quantity Q_0^a of a tradable instrument.
- All investors have to buy or sell before a given terminal time T,
- Different risk aversion parameters ϕ^a and A^a. The distribution of the risk aversion parameters is independent to anything else.
- Each investor controls its trading speed α_t^a.
- Dynamics of the the price S_t of the tradable instrument :

$$dS_t = \theta \nu_t \, dt + \sigma \, dW_t.$$

  where (W_t) is a standard B.M., $\theta > 0$ is a parameter and ν_t is the net sum of the trading speed of all investors

- In our model, the crowd impact (ν_t) is intrinsic.

The equations

- Dynamics of the the price S_t of the tradable instrument :

$$dS_t = \theta \nu_t \, dt + \sigma \, dW_t.$$

- Inventory Q_t^a of investor a :

$$dQ_t^a = \alpha_t^a \, dt,$$

- Wealth X_t^a of investor a :

$$dX_t^a = -\alpha_t^a (S_t + \kappa \cdot \alpha_t^a) \, dt, \qquad X_0^a = 0.$$

- Value function of investor a is given by :

$$V_t^a := \sup_\alpha \mathbb{E}\left[X_T^a + Q_T^a (S_T - A^a \cdot Q_T^a) - \phi^a \int_{s=t}^T (Q_s^a)^2 \, ds \Big| \mathcal{F}_t \right].$$

- Distribution of the inventories, wealth and preferences of investors : $m(t, ds, dq, da)$.
- Net trading flow ν_t :

$$\nu_t = \int_{(s,q,a)} \overline{\alpha}_t^a (s, q) \, m(t, ds, dq, da)$$

The full system

(After simplification) we obtain a Mean Field Game system of controls :

$$\left\{ \begin{array}{rcll} \partial_t u(t,a,q) - \phi^a q^2 + \dfrac{(\partial_q u(t,a,q))^2}{4\kappa} & = & -\theta q \,\nu_t & \text{in } (0,T)\times A\times\mathbb{R} \\ \partial_t m + \partial_q \left(m \frac{\partial_q u(t,a,q)}{2\kappa}\right) & = & 0 & \text{in } (0,T)\times A\times\mathbb{R} \end{array}\right.$$

where $u = u(t,a,q)$ is the value function of a typical agent,

$$\nu_t = \int_{q,a} \frac{\partial_q u(t,a,q)}{2\kappa}\, m(t,dq,da).$$

and with initial condition (for m) and terminal condition (for u) :

$$m(0,dq,da) = m_0(dq,da), \qquad u(t,a,q) = -A^a q^2 \qquad \text{in } A\times\mathbb{R}.$$

Main results (C.-Lehalle) :
— Explicit solution in the case of identical preferences,
— For general preferences, existence/uniqueness of a solution for small θ,
— Learning procedures.

Outline

Description of the model

- $\bar{m}_0$ = **initial population density** of small agents on the state space $\mathbb{R}^d$.
- $(\mu_t)_{t\in[0,T]}$ **is the density of positions and controls** of the agents. (μ_t is a Borel probability measure on $\mathbb{R}^d \times A$ with $\pi_1\sharp\mu_0 = \bar{m}_0$).
- **Dynamics of a small agent** with initial position $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$:

$$\left\{\begin{array}{l} dX_t = b(t, X_t, \alpha_t; \mu_t)dt + \sigma(t, X_t)dW_t \\ X_{t_0} = x_0 \end{array}\right.$$

where $(W_t)_{t\in[0,T]}$ is a standard $D-$dimensional Brownian Motion.

- **Value function** of the agent :

$$u(t_0, x_0) = \inf_{\alpha} \mathbb{E}\left[\int_{t_0}^{T} L(t, X_t, \alpha_t; \mu_t)\, dt + g(X_T, m_T)\right].$$

It is a viscosity solution of the HJ equation

$$\left\{\begin{array}{ll} -\partial_t u(t,x) - \mathrm{tr}(a(t,x)D^2u(t,x)) + H(t,x,Du(t,x);\mu_t) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = g(x) \quad \text{in } \mathbb{R}^d & \end{array}\right.$$

where $a = (a_{ij}) = \sigma\sigma^T$ and

$$H(t,x,p;\nu) = \sup_{\alpha}\left\{-p\cdot b(t,x,\alpha;\nu) - L(t,x,\alpha;\nu)\right\}.$$

- **Optimal feedback :** $b(t,x,\alpha^*(t,x);\mu_t) := -D_pH(t,x,Du(t,x);\mu_t)$ is the optimal drift for the agent at position x and at time t.

- Uniqueness assumption. We suppose that the map $\alpha \to b(t,x,\alpha;\nu)$ is one-to-one with a smooth inverse.
  Let $\alpha^* = \alpha^*(t,x,p;\nu)$ be the unique control such that

$$b(t,x,\alpha^*;\nu) = -D_pH(t,x,p;\nu).$$

- **At equilibrium,** the population density $m = m_t(x)$ solves the Kolmogorov equation

$$\left\{ \begin{array}{l} \partial_t m_t(x) - \sum_{i,j} \partial_{ij}(a_{ij}(t,x)m_t(x)) - \mathrm{div}\,(m_t(x)D_pH(t,x,Du(t,x);\mu_t)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathbb{R}^d \\ m_0(x) = \bar{m}_0(x) \qquad \text{in } \mathbb{R}^d \end{array}\right.$$

- **The density of the state$\times$controls** μ_t is given by the fixed-point relation :

$$\mu_t = (id, \alpha^*(t,\cdot,Du(t,\cdot);\mu_t))\,\sharp m_t.$$

To summarize, the MFG of controls takes the form :

$$\begin{cases} (i) & -\partial_t u(t,x) - \mathrm{tr}(a(t,x)D^2u(t,x)) + H(t,x,Du(t,x);\mu_t) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \\ (ii) & \partial_t m_t(x) - \sum_{i,j}\partial_{ij}(a_{ij}(t,x)m_t(x)) - \mathrm{div}\,(m_t(x)D_pH(t,x,Du(t,x);\mu_t)) = 0 \\ & \qquad\qquad \text{in } (0,T)\times\mathbb{R}^d, \\ (iii) & m_0(x) = \bar{m}_0(x),\ u(T,x) = g(x,m_T) \qquad \text{in } \mathbb{R}^d, \\ (iv) & \mu_t = (id, \alpha^*(t,\cdot,Du(t,\cdot);\mu_t))\sharp m_t \qquad \text{in } [0,T]. \end{cases}$$

Previous results :

- Gomes, Patrizi, Voskanyan (14, 16)
  Extended MFG - Existence/uniqueness of classical solutions without diffusion or ergodic.
- Carmona and Lacker (14)
  Existence/uniqueness of weak solutions when $D = d$.
- Bensoussan and Graber (16)
  Cournot-Nash equilibria - LQ framework
- Carmona and Delarue (monograph, 17)
  Second order problems - Probabilistic approach for existence/uniqueness.

New here : degenerate diffusion, regularity issues.
(classical for standard MFG systems : Lasry-Lions)

Main existence result

Theorem (C.-Lehalle)

Assume that

1. $m_0 \in L^\infty$
2. The drift has a separate form : $b(t,x,\alpha,\mu_t) = b_0(t,x,\mu_t) + b_1(t,x,\alpha)$,
3. The map $L : [0,T] \times \mathbb{R}^d \times A \times \mathcal{P}_1(\mathbb{R}^d \times A) \to \mathbb{R}$ satisfies the Lasry-Lions monotonicity condition : for any $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R}^d \times A)$ with the same first marginal,

$$\int_{\mathbb{R}^d \times A} (L(t,x,\alpha;\nu_1) - L(t,x,\alpha;\nu_2)) d(\nu_1 - \nu_2)(x,\alpha) \geq 0,$$

4. + technical regularity conditions.

Then there exists at least one solution (u,μ) to the MFG system of controls such that

- u is continuous in (t,x), Lipschitz continuous in x (uniformly with respect to t),
- m is in L^∞
- and (μ_t) is continuous from $[0,T]$ to $\mathcal{P}_1(\mathbb{R}^d \times A)$.

By a solution, we mean :

- u is satisfies equation the HJ eq. in the viscosity sense,
- m is satisfies the Kolmogorov eq. in the sense of distribution.

Ideas of proof

- As usual, by Schauder-type fixed point argument :
  — start with a family (μ_t),
  — solve HJ eq. u,
  — solve the Kolmogorov eq. $(\tilde{m}_t)$,
  — find $(\tilde{\mu}_t)$ solution of the local fixed point problem

$$\tilde{\mu}_t = (id, \alpha^*(t, \cdot, Du(t, \cdot); \tilde{\mu}_t)) \sharp \tilde{m}_t \qquad \text{in } [0, T].$$

  — show that the map $(\mu_t) \to (\tilde{\mu}_t)$ has a fixed point.

- Main issues :
  - the local fixed point and its stability
  - stability of the map Du.

The local fixed point

Lemma

Let $m \in \mathcal{P}_2(\mathbb{R}^d)$ with a bounded density and $p \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

- (Existence and uniqueness.) There exists a unique fixed point $\mu = F(p, m) \in \mathcal{P}_1(\mathbb{R}^d \times A)$ to the relation
$$\mu = (id, \alpha^*(t, \cdot, p(\cdot); \mu))\sharp m. \tag{1}$$
Moreover, there exists a constant C_0, depending only on $\|p\|_\infty$ and on the second order moment of m, such that
$$\int_{\mathbb{R}^d \times A} \left\{ |x|^2 + \delta_A(\alpha_0, \alpha) \right\} \, d\mu(x, \alpha) \leq C_0.$$
- (Stability.) Let (m_n) be a family of $\mathcal{P}_1(\mathbb{R}^d)$, with a uniformly bounded density in L^∞ and uniformly bounded second order moment, which converges in $\mathcal{P}_1(\mathbb{R}^d)$ to some m, (p_n) be a uniformly bounded family in L^∞ which converges a.e. to some p. Then $F(p_n, m_n)$ converges to $F(p, m)$ in $\mathcal{P}_1(\mathbb{R}^d \times A)$.

Uniqueness borrowed from Carmona-Delarue.

Uniform time stability stability of the gradient

Let

$$\mathcal{D} := \{p \in L^\infty(\mathbb{R}^d),\ \exists v \in W^{1,\infty}(\mathbb{R}^d),\ p = Dv,\ \|v\|_\infty \leq M,\ \|Dv\|_\infty \leq M,\ D^2 v \leq M\, I_d\}$$

endowed with the distance

$$d_{\mathcal{D}}(p_1, p_2) = \int_{\mathbb{R}^d} \frac{|p_1(x) - p_2(x)|}{(1+|x|)^{d+1}} dx \qquad \forall p_1, p_2 \in \mathcal{D}.$$

Lemma

There is a modulus ω such that, for any $(\mu_t) \in C^0([0,T], \mathcal{P}_1(\mathbb{R}^d \times A))$, the viscosity solution u to

$$\begin{cases} -\partial_t u(t,x) - \mathrm{tr}(a(t,x)D^2 u(t,x)) + H(t,x,Du(t,x);\mu_t) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

satisfies

$$d_{\mathcal{D}}(Du(t_1,\cdot), Du(t_2,\cdot)) \leq \omega(|t_1 - t_2|) \qquad \forall t_1, t_2 \in [0,T].$$

Outline

Potential for standard MFG

- Problem : find MFG equilibrium configuration as minima of an energy functional.
  Motivations : Numerical aspects, stability.

- For classical MFG systems, a potential is given by

$$\inf_{(m,\alpha)} J(m,\alpha), \qquad J(m,\alpha) := \int_0^T \int_{\mathbb{R}^d} L(x,\alpha_t(x)) m_t(dx)dt + \int_0^T \mathcal{F}(m_t)dt + \mathcal{G}(m_T),$$

where (m, v) solves

$$\partial_t m_t - \Delta m_t + \mathrm{div}(m_t \alpha_t) = 0, \qquad m_0 = \bar{m}_0.$$

- Link with the MFG system : Following Lasry-Lions, if (m, v) is a minimizer, then there exists u such that the pair (u, m) solves the MFG system

$$\begin{cases} (i) & -\partial_t u_t(x) - \Delta u_t(x) + H(t,x,Du_t(x)) = \dfrac{\delta \mathcal{F}}{\delta m}(m_t,x) \quad \text{in } (0,T)\times\mathbb{R}^d, \\ (ii) & \partial_t m_t(x) - \Delta m_t(x)) - \mathrm{div}\left(m_t(x) D_p H(t,x,Du_t(x))\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \\ (iii) & m_0(x) = \bar{m}_0(x),\ u_T(x) = \dfrac{\delta \mathcal{G}}{\delta m}(m_T,x) \quad \text{in } \mathbb{R}^d. \end{cases}$$

where $H(x,p) = \sup_{a\in\mathbb{R}^d} \{-a\cdot p - L(x,a)\}$.

Potential for MFG of controls

For MFG of controls, the potential becomes :

$$\inf_{\mu} J(\mu), \qquad J(\mu) := \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} L(x,a)\mu_t(dx,da)dt + \int_0^T \Phi(m_t,\mu_t)dt + \mathcal{G}(m_T)$$

where $\mu_t(dx,da)$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, $m_t = \pi_1 \sharp \mu_t$ and, for any $\phi \in C_c^\infty$,

$$\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} (-\partial_t \phi_t(x) - \Delta \phi_t(x) + D\phi_t(x) \cdot a)\mu_t(dx,da)dt - \int_{\mathbb{R}^d} \phi_0(x)\bar{m}_0(dx) = 0.$$

Theorem (Ben Tahar-C.)

Under “suitable" conditions, there exists at least one minimizer to J. Moreover, if μ minimizes J, then there exists (u,m) such that (u,m,μ) solves the extended MFG system

$$\left\{ \begin{array}{ll} (i) & -\partial_t u_t(x) - \Delta u_t(x) + \mathbf{H}(x, Du_t(x); m_t, \mu_t) = \dfrac{\delta \Phi}{\delta m}(m_t, \mu_t, x), \\ (ii) & \partial_t m_t(x) - \Delta m_t(x) - \mathrm{div}(m_t(x) D_p \mathbf{H}(x, Du_t(t,x); m_t, \mu_t)) = 0, \\ (iii) & \mu_t = (id, -D_p \mathbf{H}(\cdot, Du_t(\cdot); m_t, \mu_t))\sharp m_t, \\ (iv) & m_0 = \bar{m}_0,\ u_T = \dfrac{\delta \mathcal{G}}{\delta m}(m_T, \cdot). \end{array} \right.$$

where $$\mathbf{H}(x,p;m,\mu) := \sup_{a \in \mathbb{R}^d} \left\{ -a \cdot p - L(x,a) - \frac{\delta \Phi}{\delta \mu}(m,\mu,x,a) \right\}.$$

Conclusion and open problems

We have developed

- For a simple model of trade crowding : well-posedness, learning,
- General existence result for MFG of controls with degenerate diffusion,
- Conditions under which the MFG of controls is a potential game.

Open problems :

- Use of the solution of the MFG system of controls in problems with N players,
- Learning procedures.

Assumptions

1. $g : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times D}$ are smooth and bounded,
2. The drift has a separate form : $b(t, x, \alpha, \mu_t) = b_0(t, x, \mu_t) + b_1(t, x, \alpha)$,
3. The map $L : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}_1(\mathbb{R}^d \times A) \to \mathbb{R}$ satisfies the Lasry-Lions monotonicity condition : for any $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R}^d \times A)$ with the same first marginal,

$$\int_{\mathbb{R}^d \times A} (L(t, x, \alpha; \nu_1) - L(t, x, \alpha; \nu_2)) d(\nu_1 - \nu_2)(x, \alpha) \geq 0,$$

4. The map $\alpha^* : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d \times A) \to \mathbb{R}$ is continuous, with a linear growth : for any $L > 0$, there exists $C_L > 0$ such that

$$\delta_A(\alpha_0, \alpha^*(t, x, p; \nu)) \leq C_L(|x| + 1) \qquad \forall (t, x, p, \nu) \text{ with } |p| \leq L,$$

(where α_0 is a fixed element of A).

5. The Hamiltonian $H : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d \times A) \to \mathbb{R}$ is continuous ; H is bounded in C^2 in (x, p) uniformly with respect to (t, ν), and convex in p.
6. The initial measure $\bar{m}_0$ is a continuous probability density on $\mathbb{R}^d$ with a finite second order moment.