Viability, arbitrage and preferences

Matteo Burzoni
joint work with F. Riedel and H. M. Soner

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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MOTIVATION

Viability in Kreps [78], Harrison-Kreps [79]

Suppose we are given a price system (M, π) as

- a linear space $M \subset X$ of marketed claims,
- a linear pricing rule π defined on M.

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Q: Does this follow from some economic principle?

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- satisfies the budget constraint: $\pi(m^*) \leq 0$.
- is optimal: $m^* \succeq m$ for all $m \in M$ with $\pi(m) \leq 0$

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Under some restrictions on $\preceq$ (conceivable agents) we have the following

Theorem (K78, HK79)

$$\textit{viability} \iff \pi \textit{ admits an extension to } X$$

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$$\exists K \subset X \quad \text{s.t.} \quad x + k \succ x \qquad \forall x \in X,\ k \in K,$$

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- it determines the continuity requirement for $\preceq$ (level sets are L^2-closed).

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Theorem (HK79)

viability $\Longleftrightarrow$ $\mathbb{P}$ *admits an equivalent martingale measure*

AIM AND GOALS

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To study:

- viability in a general framework,
- connection with no arbitrage,
- connection with extendability.

Outline

- A general framework.
- Arbitrage and Viability.
- Characterization in terms of sublinear expectations.

THE SETUP

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  - $Z \in \mathcal{H}$ is *negligible* if $Z \sim \mathbf{0}$ (i.e $Z \geq 0$ and $Z \leq 0$);
  - $P \in \mathcal{H}$ is *non-negative* if $P \geq \mathbf{0}$ and *positive* if $P > \mathbf{0}$.

Notation: $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{P}^+$ respectively.

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- The set of contracts achievable with zero initial cost or in short, *achievable contracts* is a given convex cone $\mathcal{I}$.

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$$F \preceq X \text{ and } F \preceq Y \quad \Rightarrow F \preceq \lambda X + (1-\lambda) Y, \quad \forall \lambda \in [0,1];$$

- $\preceq$ is *weakly continuous*, i.e., for every sequence $\{c_n\} \subset \mathbb{R}_+$ with $c_n \downarrow 0$ we have

$$X - \mathbf{c}_n \preceq Y, \ \forall n \in \mathbb{N} \quad \Rightarrow \quad X \preceq Y, \qquad X, Y \in \mathcal{H}.$$

ARBITRAGE AND VIABILITY

Definition

Let Θ be a financial market and $\mathcal{R} \subset \mathcal{P}^+$ a class of *relevant contracts*.

▷ We say that a sequence of achievable contracts $\{\ell^n\}_{n=1}^{\infty} \subset \mathcal{I}$ is a *free lunch with vanishing risk*, if there exists a relevant contract $R^* \in \mathcal{R}$ and a sequence $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ with $c_n \downarrow 0$ satisfying,

$$\mathbf{c}_n + \ell^n \geq R^*, \quad n = 1, 2, \ldots$$

write $NA_s(\Theta, \mathcal{R})$ when $(\Theta, \mathcal{R})$ has no free lunches with vanishing risk.

Viability

Economically, a price system in a financial market is viable if it can be derived from an economic equilibrium in which agents have preferences from $\mathcal{A}$.

Equilibrium in this context would mean that one can find a best net trade $\ell^* \in \mathcal{I}$ so that by adding an achievable contract with zero cost, $\ell \in \mathcal{I}$, to ℓ^* we cannot obtain a preferable contract.

The existence of such an optimal contract ℓ^* is a necessary condition for equilibrium.

Definition

Let $(\Theta, \mathcal{R})$ be a financial market. We say that $(\Theta, \mathcal{R})$ is *viable*, if there exists $\preceq' \in \mathcal{A}$ and a net trade vector $\ell^* \in \mathcal{I}$ satisfying

$$\begin{aligned} \ell + X \preceq' \ell^* + X, &\qquad \forall \ell \in \mathcal{I},\ X \in \mathcal{H}. \\ \ell^* - R \prec' \ell^*, &\qquad \forall R \in \mathcal{R}. \end{aligned}$$

The first is an equilibrium condition. In particular, for $X = 0$,

$$\ell \preceq' \ell^*, \quad \forall\, \ell \in \mathcal{I}.$$

The second replaces and weaken the classical monotonicity condition assumed in Kreps, Harrison and Kreps ['79]. Strict monotonicity is required only at the optimal.

Main Result

Theorem

Let $(\Theta, \mathcal{R})$ be a financial market. The following are equivalent:

1. *$(\Theta, \mathcal{R})$ is viable;*
2. *$(\Theta, \mathcal{R})$ has no free lunch with vanishing risk.*

Sketch of the proof

We introduce the superhedgding functional:

$$\mathcal{D}(X) := \inf \{ \; c \in \mathbb{R} \; : \; \exists \; \ell \in \mathcal{I} \text{ so that } c + \ell \geq X \; \}, \quad X \in \mathcal{H}$$

Proposition

The financial market satisfies $NA_s(\Theta)$ if and only if $\mathcal{D}(p) > 0$, $\forall \; p \in \mathcal{R}$.

Sketch of the proof

$NA_s(\Theta) \Rightarrow$ Viability.
Define the utility function $U(X) := -\mathcal{D}(-X)$ for $X \in \mathcal{H}$.

Define $\preceq$ on $\mathcal{H}$ by

$$X \preceq Y \quad \Leftrightarrow \quad U(X) \leq U(Y).$$

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Moreover, if $Y - \mathbf{c}_n \preceq X$ with $c_n \downarrow 0$. Then,

$$U(Y) - c_n = U(Y - \mathbf{c}_n) \leq U(X) \ \forall n \quad \Rightarrow \quad U(Y) \leq U(X) \quad \Rightarrow \quad Y \preceq X.$$

Hence, $\preceq$ is weakly continuous. This shows that $\preceq \in \mathcal{A}$.

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$NA_s(\Theta) \Rightarrow$ Viability.

Next we show viability. For any $X \in \mathcal{H}, \ell \in \mathcal{I}$,

$$U(X+\ell) = -\mathcal{D}(-[X+\ell]) \leq -\mathcal{D}(-[X+\ell]+\ell) = -\mathcal{D}(-X) = U(X).$$

Hence, $X + \ell \preceq X$ for any $X \in \mathcal{H}$ and $\ell \in \mathcal{I}$.

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Hence, $X + \ell \preceq X$ for any $X \in \mathcal{H}$ and $\ell \in \mathcal{I}$.

Also $NA_s(\Theta)$ implies that $\mathcal{D}(R) > 0$ and $\mathcal{D}(\mathbf{0}) = 0$. Therefore,

$$U(-R) = -\mathcal{D}(R) < 0 = U(\mathbf{0}) \quad \Rightarrow \quad -R \prec \mathbf{0}.$$

We conclude that $(\Theta, \mathcal{R})$ is viable.

CHARACTERIZATION IN TERMS OF SUBLINEAR EXPECTATIONS

Sublinear martingale expectations

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We denote by $\mathcal{M}(\Theta, \mathcal{R})$ the class of sublinear expectations, which satisfies the properties listed above. $\mathcal{M}^c(\Theta, \mathcal{R})$ those which are, in addition, coherent.

Characterization

Suppose for the moment $\mathcal{H} = \mathcal{B}_b$.

Theorem

For a financial market $\Theta = (\preceq, \mathcal{I}, \mathcal{R})$, the following are equivalent:

1. *The financial market is viable.*
2. *$NA_s(\Theta)$ holds true.*
3. *The set $\mathcal{M}(\Theta, \mathcal{R})$ is non-empty.*
4. *There exists a convex set of linear functionals $\mathcal{Q}(\Theta) \subset ba$ satisfying*
   - $\varphi(\Omega) = 1$,
   - $\varphi(P) \geq 0$ *for every* $P \in \mathcal{P}$.
   - $\varphi(\ell) \leq 0$ *for every* $\ell \in \mathcal{I}$,
   - *for any* $R \in \mathcal{R}$, *there exists* $\varphi_R \in \mathcal{Q}$ *such that* $\varphi_R(R) > 0$.

EXAMPLES

Probabilistic framework

Let $\mathcal{F}$ be a sigma algebra on Ω and $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.

- Weak Market Efficient Hypothesis

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{E}^{\mathbb{P}}[X] \leq \mathbb{E}^{\mathbb{P}}[Y].$$

$\mathcal{Z}$ is the set of all functions with mean zero. Typically $\mathcal{R} = \mathcal{P}^+$.

- Strong Market Efficient Hypothesis

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1.$$

$\mathcal{Z}$ is the set of $\mathbb{P}$-a.s. zero functions. Typically $\mathcal{R} = \mathcal{P}^+$.

Market Efficient Hypothesis under Ambiguity

Let $\mathcal{M}$ be a given set of probability measures on $(\Omega, \mathcal{F})$. Define

$$\mathcal{E}_{\mathcal{M}}(X) := \inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}}[X], \quad X \in \mathcal{H}.$$

We can extend the previous examples by incorporating ambiguity through the nonlinear expectation $\mathcal{E}_{\mathcal{M}}$. In particular, for the Strong Market Efficient Hypothesis under Ambiguity,

$$X \leq Y \quad \Leftrightarrow \quad 0 \leq \mathcal{E}_{\mathcal{M}}[Y - X].$$

$\mathcal{Z}$ is the set of $\mathcal{M}$-q.s. zero functions and, typically,
$\mathcal{R} := \mathcal{P}^{+} = \{P \geq 0 \ \mathcal{M} - q.s., \ \mathbb{P}(P > 0) > 0 \text{ for some } \mathbb{P} \in \mathcal{M}\}.$

Pointwise framework

In the following examples we let Ω be a metric space. We say $X \preceq Y$ if

$$\inf_{\Omega} X \leq \inf_{\Omega} Y \ ,$$

which implies $\mathcal{Z} = \{0\}$. Different choices for $\mathcal{R}$ leads to different notion of arbitrage.

1. $\mathcal{R} := \{P \in \mathcal{P} \ : \ \exists \omega_0 \in \Omega$ such that $P(\omega_0) > 0 \ \}$. *One point arb.*
2. $\mathcal{R} := \{P \in C_b(\Omega) \ : \ \exists \omega_0 \in \Omega$ such that $P(\omega_0) > 0 \ \}$. *Open arb.*
3. $\mathcal{R} = \{P \in \mathcal{P} \ : \ \exists c \in (0, \infty)$ such that $P \geq_{\Omega} c \ \}$. *Uniform arb.*

Conclusions

- We have provided a general framework to study arbitrage and viability.
- This framework allows for probabilistic and non-probabilistic descriptions. Continuous or discrete time markets. General set of investement opportunities.
- A market with no free lunch is viable and viceversa.
- Pricing mechanism arising from the market might be in *ba*.
- Pricing rules with full support are in general non-linear.

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Thank you for your kind attention.