Characterization of a Wasserstein type distance in terms of a stochastic control problem

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INTRODUCTION

Motivated by ROBUST CALIBRATION PROBLEMS FOR DIFFUSION TYPE MODELS we introduce a new distance on the set of probability distributions which are solutions to martingale problems. Like the classical Wasserstein distance, THIS NEW DISTANCE METRIZES THE WEAK TOPOLOGY.
This new distance is the value function of a stochastic control problem. We prove that THE CORRESPONDING HAMILTON-JACOBI-BELLMAN EQUATION HAS A SMOOTH SOLUTION.
This allows one to obtain numerical evaluations.
We finally exhibit an optimal coupling measure.

OUTLINE

$\Omega_d = L_2([0,T], \boldsymbol{R}^d)$ is a Polish space. The Wasserstein distance $\mathcal{W}^2$ metrizes the weak topology on the set of measures $\mathbb{P}$ such that $\mathbb{E}^{\mathbb{P}} \int_0^T |\omega_s|^2 \, ds < \infty$.

$$\mathcal{W}^2(\mathbb{P};\overline{\mathbb{P}}) := \left\{ \inf_{\pi \in \Pi(\mathbb{P};\overline{\mathbb{P}})} \int_{\Omega_{2d}} \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \pi(d\omega, d\overline{\omega}) \right\}^{\frac{1}{2}}, \qquad (1)$$

π has marginal distributions $\mathbb{P}$ and $\overline{\mathbb{P}}$.
Unfortunately numerical computation is impossible.

DEFINITION

$$\mathbf{P} = \{\mathbb{P}_x^{\mu,\sigma}, \mu, \sigma \text{ Lipschitz}, \sigma \text{ strongly elliptic}, \ x \in \boldsymbol{R}^d\}$$

where $\mathbb{P}_x^{\mu,\sigma}$ is the probability distribution of the unique strong solution to the stochastic differential equation with coefficients μ and σ and initial condition x.

Definition

Given two probability measures $\mathbb{P}_x^{\mu,\sigma}$ and $\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}}$ belonging to **P**, let $\widetilde{\Pi}(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}})$ be the set of the probability distribution $\widetilde{\mathbb{P}}$ of $(X^{\mathcal{C}},\overline{X})$ solution to the following system of SDEs:

$$\begin{cases} dX_s^{\mathcal{C}} = \mu(X_s^{\mathcal{C}})\,ds + \sigma(X_s^{\mathcal{C}})\,(\mathcal{C}_s\,d\overline{W}_s + \mathcal{D}_s\,dW_s), \\ d\overline{X}_s = \overline{\mu}(\overline{X}_s)\,ds + \overline{\sigma}(\overline{X}_s)\,d\overline{W}_s, \end{cases} \tag{2}$$

with initial condition $(x,\overline{x})$, with (C_s) predictable with values in correlation matrices and $\mathcal{D}_s = \sqrt{\mathrm{Id}_d - \mathcal{C}_s\,\mathcal{C}_s^{\intercal}}$ for any $0 \le s \le T$.

DEFINITION OF $\widetilde{\mathcal{W}}^2$

DEFINITION

$$\widetilde{\mathcal{W}}^2(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}}) := \left\{ \inf_{\widetilde{\mathbb{P}}\in\widetilde{\Pi}(\mathbb{P}_x^{\mu,\sigma};\mathbb{P}_{\overline{x}}^{\overline{\mu},\overline{\sigma}})} \int_\Omega \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \widetilde{\mathbb{P}}(d\omega, d\overline{\omega}) \right\}^{\frac{1}{2}}. \tag{3}$$

PROPOSITION

$\widetilde{\mathcal{W}}^2$ metrizes the weak topology on the set of probability distributions $\mathbb{P}_x^{\mu,\sigma}$ with coefficients μ and σ uniformy Lipschitz bounded and x in a compact set.

OUTLINE

Interpretation of $\widetilde{\mathcal{W}}^2$

The corresponding Hamilton–Jacobi–Bellman equation is the following:

$$\begin{cases} \partial_t V(t,x,\overline{x}) + \mathcal{L}V(t,x,\overline{x}) + H(t,x,\overline{x},V) = -|x-\overline{x}|^2, \ 0 \le t < T, \\ V(T,x,\overline{x}) = 0, \end{cases} \tag{4}$$

$$\begin{aligned} \mathcal{L}V(t,x,\overline{x}) := & \sum_{i=1}^{d} \mu_i(x)\partial_{x_i} V(t,x,\overline{x}) + \sum_{i=1}^{d} \overline{\mu}_i(\overline{x})\partial_{\overline{x}_i} V(t,x,\overline{x}) \\ & + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\intercal(x))^{ij}\partial^2_{x_i,x_j} V(t,x,\overline{x}) + \frac{1}{2}\sum_{i,j=1}^{d} (\overline{\sigma}(\overline{x})\overline{\sigma}(\overline{x})^\intercal)^{ij}\partial^2_{\overline{x}_i,\overline{x}_j} V(t,x,\overline{x}) \end{aligned}$$

$$\text{and } \ H(t,x,\overline{x},V) := \min_{C\in\mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x)C\overline{\sigma}(\overline{x})^\intercal)^{ij}\partial^2_{x_i,\overline{x}_j} V(t,x,\overline{x}).$$

One dimensional case

Consider the family of stochastic differential equations

$$\forall t \leq \theta \leq T, \qquad X^*_\theta = x + \int_t^\theta \mu(X^*_s)\, ds + \int_t^\theta \sigma(X^*_s)\, d\overline{W}_s.$$
$$\overline{X}_\theta = \overline{x} + \int_t^\theta \overline{\mu}(\overline{X}_s)\, ds + \int_t^\theta \overline{\sigma}(\overline{X}_s)\, d\overline{W}_s.$$

consider the function

$$V^*(t,x,\overline{x}) := \mathbb{E}\int_t^T \left(X^*_\theta(t,x) - \overline{X}_\theta(t,\overline{x})\right)^2 d\theta.$$

This function is the unique classical solution to the parabolic PDE

$$\begin{cases} \frac{\partial V^*}{\partial t} + \mu(x)\,\frac{\partial V^*}{\partial x} + \overline{\mu}(\overline{x})\,\frac{\partial V^*}{\partial \overline{x}} + \frac{1}{2}\sigma^2(x)\,\frac{\partial^2 V^*}{\partial x^2} + \frac{1}{2}\overline{\sigma}^2(\overline{x})\,\frac{\partial^2 V^*}{\partial \overline{x}^2} \\ \qquad +\sigma(x)\overline{\sigma}(\overline{x})\frac{\partial^2 V^*}{\partial x \partial \overline{x}} = -(x-\overline{x})^2, \\ V^*(T,x,\overline{x}) = 0. \end{cases}$$

ONE DIMENSIONAL CASE

$$\frac{\partial^2 V^*}{\partial x \partial \overline{x}}(t,x,\overline{x}) = -2\int_t^T \mathbb{E}\left[\frac{d}{dx}X_s^*(t,x)\ \frac{d}{d\overline{x}}\overline{X}_s(\overline{x})\right]\ ds.$$

The derivative of the flow is a stochastic exponential. It follows that

$$\forall t,x,\overline{x},\ \ \frac{\partial^2 V^*}{\partial x \partial \overline{x}}(t,x,\overline{x}) < 0.$$

Therefore we have exhibited a classical solution $V^*(t,x,\overline{x})$ to the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial V}{\partial t} + \mu(x)\ \frac{\partial V}{\partial x} + \overline{\mu}(\overline{x})\ \frac{\partial V}{\partial \overline{x}} + \frac{1}{2}\sigma^2(x)\ \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\overline{\sigma}^2(\overline{x})\ \frac{\partial^2 V}{\partial \overline{x}^2} \\ \qquad + \min_{C\in[-1,1]}\left(C\ \sigma(x)\ \overline{\sigma}(\overline{x})\ \frac{\partial^2 V}{\partial x \partial \overline{x}}\right) = -(x-\overline{x})^2, \\ V(T,x,\overline{x}) = 0. \end{cases}$$

Multidimensional case: Existence of a regular solution to the HJB equation

Theorem

Suppose:

(I) *The functions μ, $\overline{\mu}$, σ and $\overline{\sigma}$ are in the Hölder space $\mathcal{C}^{1+\alpha}(\boldsymbol{R}^d)$ with* $0 < \alpha \leq 1$.

(II) *The matrix-valued functions* $a(x) := \sigma(x)\sigma(x)^\intercal$ *and* $\overline{a}(x) := \overline{\sigma}(x)\overline{\sigma}(x)^\intercal$ *satisfy the strong ellipticity condition*

$$\exists \lambda > 0,\ \forall \xi,\ \overline{\xi},\ x,\ \sum_{i,j=1}^{d} a^{ij}(x)\xi^i\xi^j + \sum_{i,j=1}^{d} \overline{a}^{ij}(x)\overline{\xi}^i\overline{\xi}^j \geq \lambda|(\xi|^2 + |\overline{\xi}|^2).$$

Then there exists a solution V to the HJB equation (4), $V \in \mathcal{C}^{1,2}([0,T] \times \boldsymbol{R}^d)$ *such that $\partial_t V$, $\partial^2_{x_i,\bar{x}_j} V$,... are $(\frac{\alpha}{2}, \alpha)$ Hölder.*

Approximation among diffusion laws

Recall that $H(t,x,\bar{x},V) = \min_{C\in\mathbf{C}_d} \sum_{i,j=1}^{d} (\sigma(x)C\overline{\sigma}(\bar{x})^\intercal)^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x})$

Proposition

For every $\epsilon > 0$ there is a continuous map $C_\epsilon(s,x,\bar{x})$ taking values in the set of correlation matrices which is pointwise ϵ-optimal: $\forall s,x,\bar{x}$,

$$H(t,x,\bar{x},V) \leq \sum_{i,j=1}^{d} (\sigma(x)C_\epsilon(s,x,\bar{x})\overline{\sigma}(\bar{x})^\intercal)^{ij}\partial^2_{x_i,\bar{x}_j}V(t,x,\bar{x}) \leq H(t,x,\bar{x},V) + \epsilon. \tag{5}$$

There exists a sequence $(\mathbb{P}^{\mu^m,\sigma^m})$ of solutions to martingale problems with continuous Markovian coefficients μ^m and σ^m such that

$$\int_\Omega \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \mathbb{P}^{\mu^m,\sigma^m}(d\omega, d\overline{\omega})$$

converges to $\widetilde{\mathcal{W}}^2(\mathbb{P}^{\mu,\sigma};\mathbb{P}^{\overline{\mu},\overline{\sigma}})$.

Existence of an optimal coupling measure

Theorem

There exist a predictable process C^* and an adapted and continuous solution on $[0,T]$ to the system

$$\begin{cases} X_t^* = x + \int_0^t \mu(X_s^*)\,ds + \int_0^t \sigma(X_s^*)\,C_s^*\,d\overline{W}_s + \int_0^t \sigma(X_s^*)\,D_s^*\,dW_s, \\ \overline{X}_t = \bar{x} + \int_0^t \overline{\mu}(\overline{X}_s)\,ds + \int_0^t \overline{\sigma}(\overline{X}_s)\,d\overline{W}_s, \\ C_s^* \in \underset{C \in \mathbf{C}_d}{\arg\min} \sum_{i,j=1}^d \left(\sigma(X_s^*)\,C\,\overline{\sigma}(\overline{X}_s)^T\right)^{ij} \partial^2_{x_i \bar{x}_j} V(s, X_s^*, \overline{X}_s)\right), \\ D_s^* = \sqrt{\mathrm{Id}_d - C_s^*\,C_s^{*\mathsf{T}}} \end{cases} \tag{6}$$

which satisfies

$$V(0,x,\bar{x}) = \mathbb{E}\int_0^T |X_t^* - \overline{X}_t|^2 dt = \widetilde{\mathcal{W}}^2(\mathbb{P}_x^{\mu,\sigma}; \mathbb{P}_{\bar{x}}^{\overline{\mu},\overline{\sigma}}).$$