Superhedging with transient impact

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Advances in Stochastic Analysis for Risk Modeling

Outline

- Model of price impact
  - Price process specification
  - Deriving “the” wealth process
- Superhedging of European options by a large trader
  - Stochastic target approach
  - Pricing PDE
  - Dynamic programming principle, in special coordinates

Multiplicative price impact model

- One risky asset, interest rate $= 0$;
- *Unaffected (fundamental) price process*

$$\mathrm{d}\overline{S}_t = \mu \overline{S}_t \,\mathrm{d}t + \sigma \overline{S}_t \,\mathrm{d}W_t, \quad \overline{S}_0 = s.$$

- **Price impact by a large trader**
  - $(\Theta_t)_{t\geq 0}$ - holdings in the risky asset, (bounded) càdlàg process;
  - *market impact process*: for Lipschitz $h : \mathbb{R} \to \mathbb{R}$, $h(0) = 0$, $h(x)\mathrm{sgn}(x) \geq 0$,

$$\mathrm{d}Y_t^\Theta = -h(Y_t^\Theta)\,\mathrm{d}t + \mathrm{d}\Theta_t, \quad Y_{0-} = y \in \mathbb{R};$$

  - **Current 'affected' price of the risky asset**

$$S_t = S_t^\Theta := f(Y_t^\Theta)\overline{S}_t$$

with price impact function $f(y) := \exp\left(\int_0^y \lambda(u)\,\mathrm{d}u\right)$ for $\lambda : \mathbb{R} \to [0, \infty)$

- **Key features:** transient impact, positive prices.

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Proceeds from trading

For continuous finite-variation strategies

- Consider a block trade at time t: $\Delta\Theta_t \neq 0$; price of the asset before the trade $= f(Y^{\Theta}_{t-})\overline{S}_t$, after the trade $= f(Y^{\Theta}_{t-} + \Delta\Theta_t)\overline{S}_t$.
- Issue: $f(Y^{\Theta}_{t-})\overline{S}_t \neq f(Y^{\Theta}_{t})\overline{S}_t$
- **How to define proceeds for block trades or càdlàg strategies?**

Starting point: For a **continuous finite variation** Θ, proceeds from trading are

$$L_T(\Theta) := -\int_0^T \underbrace{f(Y^{\Theta}_u)\overline{S}_u}_{S_u} \,\mathrm{d}\Theta_u.$$

Goal: extend definition of L **continuously** to semimartingales/càdlàg processes

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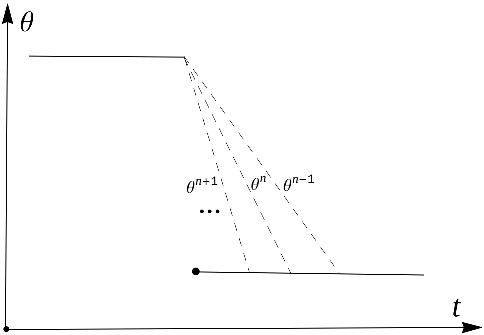

Figure: A block trade approximated by a sequence of abs. continuous trades.

Remark: The **Skorokhod** M_1 **topology** captures this approximation.

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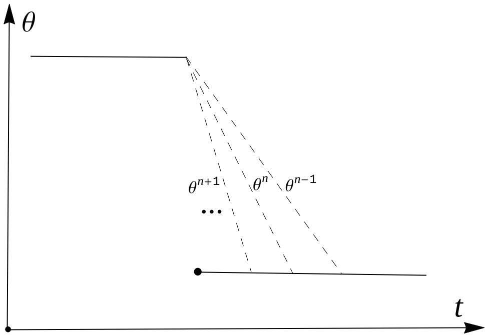

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Proceeds for semimartingale strategies

- Proceeds from bounded semimartingale strategies Θ

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$$- \sum_{\substack{\Delta\Theta_t \neq 0 \\ t \leq \cdot}} \overline{S}_t \Big(\int_0^{\Delta\Theta_t} f(Y^{\Theta}_{t-} + x)\,\mathrm{d}x - f(Y^{\Theta}_{t-})\Delta\Theta_t \Big)$$

- Proceeds from a block trade of size $\Delta\Theta_t$ are $-\overline{S}_t \int_0^{\Delta\Theta_t} f(Y^{\Theta}_{t-} + x)\,\mathrm{d}x$
- Self-financing condition for portfolios (β, Θ): $\beta_t = \beta_{0-} + L_t(\Theta)$, $t \geq 0$;
- *Instantaneous liquidation* wealth process

$$V^{\mathrm{liq}}_t(\Theta) := \beta_t + \overline{S}_t \int_0^{\Theta_t} f(Y^{\Theta}_t - x)\,\mathrm{d}x.$$

D. Becherer, T. B., P. Frentrup, *Stability for gains from large investors' strategies in M1/J1 topologies*, arXiv:1701.02167 (2017)

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Superhedging in illiquid markets

Definition

European option is specified by the maturity T and the payoff (g_0, g_1) with measurable $g_0, g_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $g_{0/1}(S, Y)$ represents the cash/physical settlement, where S is the (affected) price and Y is the market impact, evaluated at T.

- Admissible self-financing (β, Θ) is a **superhedging strategy** for the European claim with maturity T and payoff (g_0, g_1) if
  - $\Theta_{0-} = 0$
  - $\Theta_T = g_1(S_T^\Theta, Y_T^\Theta)$ and $\beta_T \geq g_0(S_T^\Theta, Y_T^\Theta)$
- Superhedging price is

$$\inf\{\beta_{0-} \mid \text{self-financing superhedging } (\beta, \Theta) \text{ exists}\}$$

Remark: Initial and terminal impact are included.

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Admissible strategies

- Trading strategies Γ_N for $N \geq 1$:

$$\Theta_t = \Theta_{0-} + \int_0^t a_s \,\mathrm{d}s + \int_0^t b_s \,\mathrm{d}W_s + \sum_{i=1}^{N} \delta_i \mathbb{1}_{\{\tau_i \leq t\}},$$

  where $\tau_1, \ldots, \tau_n$ are stopping times, $\delta_i \in [-N, N]$ is $\mathcal{F}_{\tau_i}$-measurable, a and b and Θ are predictable and bounded by N

- **Admissible strategies:** $\Gamma := \bigcup_{N \geq 1} \Gamma_N$.

The superhedging problem as stochastic target problem

- **State process**:

$$Z^{t,z,\gamma} := (S^{t,z,\gamma}, Y^{t,z,\gamma}, \Theta^{t,z,\gamma}, V^{\mathrm{liq},t,z,\gamma}),$$

where $(t,z) = (t,s,y,\theta,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathcal{K} \times \mathbb{R}$ and $\gamma \in \Gamma$.

- **Target set** for the state process:

$$\mathfrak{G} := \{(s,y,\theta,v) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \; : \; \theta = g_1(s,y), \; v - s\frac{F(y)-F(y-\theta)}{f(y)} \geq g_0(s,y)\}.$$

- **Superhedging strategies:**

$$\mathcal{G}(t,s,y,\theta,v) := \{\gamma \in \Gamma \; : \; Z_T^{t,s,y,\theta,v,\gamma} \in \mathfrak{G}\}.$$

- **Minimal superreplication price**:

$$w(t,s,y) := \inf\{v \; : \; \mathcal{G}(t,s,y,0,v) \neq \emptyset\}.$$

Pricing pde

Theorem (Pricing equation)

Let $F(x) = \int_0^x f(u)\,\mathrm{d}u$ *and*

$$G(s,y) := \inf\left\{ g_0\left(s\frac{f(y+\theta)}{f(y)}, y+\theta\right) + s\frac{F(y+\theta)-F(y)}{f(y)} \mid \theta = g_1\left(s\frac{f(y+\theta)}{f(y)}, y+\theta\right)\right\}.$$

Under assumptions on $f, h, (g_0, g_1)$, *the minimal superreplication price* w *is the unique viscosity solution of*

$$-w_t - \frac{1}{2}\sigma^2 s^2 w_{SS} + \tilde{h}(t,s,y)\left(w_Y + s\lambda(y)w_S + s - s\frac{\tilde{f}(t,s,y)}{f(y)}\right) = 0,$$

with boundary condition $w(T,\cdot) = G(\cdot)$, *where for* $(t,s,y) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}$

$$\tilde{h}(t,s,y) := h \circ F^{-1}\big(f(y)w_S(t,s,y) + F(y)\big),$$
$$\tilde{f}(t,s,y) := f \circ F^{-1}\big(f(y)w_S(t,s,y) + F(y)\big).$$

Remarks

- If $h \equiv 0$ (permanent impact), minimal superreplication price for large investor is the Black-Scholes price for the payoff G.

- If $f(x) = \exp(\lambda x)$ with constant λ, then for typical options pricing pde simplifies to Black-Scholes pde
- For sufficiently regular solutions of the pricing pde, optimal replicating strategies can be constructed: with $\mathcal{Y}^\Theta = Y^\Theta - \Theta$, for $t \in [0, T)$

$$\Theta_t = F^{-1}\left(f(\mathcal{Y}_t^\Theta) w_S(t, \mathcal{S}(S_t, Y_t^\Theta, \Theta_t), \mathcal{Y}_t^\Theta) + F(\mathcal{Y}_t^\Theta)\right) - \mathcal{Y}_t^\Theta$$

- For *covered options* (no initial and terminal impact), the pricing pde is of qualitatively different nature (*gamma constraints* appear).

Numerical example - call option with physical delivery

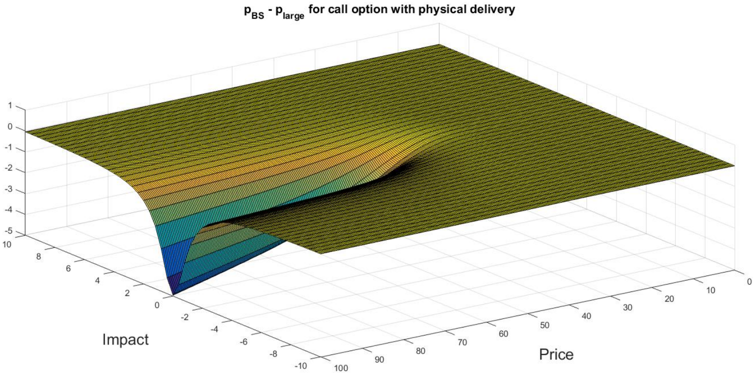

Figure: Parameters: $f(x) = 1 + \arctan(x)/10$, $K = 50$, $T = 0.5$, $\sigma = 0.3$, $h(y) = y$

How to derive the pde: New coordinates

Recall: $w(t, s, y) := \inf\{v \ : \ \mathcal{G}(t, s, y, 0, v) \neq \emptyset\}$

Problem: Dynamic programming principle for w does not hold per se: we have assumed zero initial holdings.

Way out: (adapted from Bouchard, Loeper and Zhou, F&S 2016)

effective price process $\mathcal{S}(S^\Theta, Y^\Theta, \Theta) := \overline{S} f(Y^\Theta - \Theta)$

effective impact process $\mathcal{Y}^\Theta := Y^\Theta - \Theta$

Interpretation:

- $\mathcal{S}(s, y, \theta)$ is the price of the asset that would prevail after θ risky assets have been sold with a block trade, with s (resp. y) being the price (resp. impact) just before the trade.

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Geometric Dynamic Programming Principle

Proposition (Geometric DPP - Part 1)

Fix $(t, s, y, v) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$.

(1) *Let* $v > w(t, s, y)$. *Then there exists* $\gamma \in \Gamma$ *and* $\theta \in \mathcal{K}$ *such that*

$$V_\tau^{liq,t,z,\gamma} \geq w(\tau, \mathcal{S}(S_\tau^{t,z,\gamma}, Y_\tau^{t,z,\gamma}, \Theta_\tau^{t,z,\gamma}), Y_\tau^{t,z,\gamma} - \Theta_\tau^{t,z,\gamma})$$

for all stopping times $\tau \geq t$, *where* $z = (\mathcal{S}(s, y, -\theta), y + \theta, \theta, v)$.

Conclusion

- Multiplicative price impact does not yield negative prices
- Impact is transient
- Wealth process derived via continuity arguments
- Pricing equation for European options is non-linear Black-Scholes pde where non-linearity is mainly due to resilience