Good Deal Hedging and Valuation under Combined Uncertainty about Drift & Volatility: a 2nd-order-BSDE approach

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Motivating example

- How to hedge and value a **put option** $X = (\mathcal{K} - H_T)^+$ on a **non-traded asset**

$$dH_t = H_t\big(\gamma dt + \beta(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2)\big)$$

if there is only a **correlated** tradable asset

$$dS_t = S_t \sigma^S (\xi dt + dB_t^1)$$

available for **partial hedging** : ?

- non-perfect correlation: $-1 < \rho < 1$.

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available for **partial hedging** : ?

- non-perfect correlation: $-1 < \rho < 1$.
- $\rightsquigarrow$ Superreplication (a.s.-hedging) is prohibitively expensive! Hedge would be extreme but trivial (don't trade even if $\rho = 99\%$)! **Uncertainty on drift&volatilitiy** matters for any alternative...

Aspect I: incomplete vs complete

- Complete market: unique price obtained by replication

$$X = \underbrace{E^Q[X]}_{\text{replication cost}} + \underbrace{\int_0^T \phi_t dS_t}_{\text{gain/loss from trading}} \quad \text{a.s.,} \quad \text{for } \mathcal{M}^e(S) = \{Q\}.$$

- Incomplete market: infinitely many martingale measures $\mathcal{M}^e(S)$, upper NA-bound is seller's super-replicating price

$$\sup_{Q \in \mathcal{M}^e(S)} E^Q[X] = \inf \left\{ m : \exists \phi \text{ s.t. } m + \int_0^T \phi_t dS_t \geq X \ P\text{-a.s.} \right\}.$$

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- **But:** Superreplication is too expensive, a.s.-hedge is too extreme !
- **Aim:** less expensive valuations, less extreme than a.s.-hedging. How? Exclude not just arbitrage but also "too good deals"
- $\rightsquigarrow$ hedging error distribution matters, hence **uncertainty** on P!

Ansatz for no-good-deal valuation and hedging

- **Valuation:** Use only subset $\mathcal{Q}^{\mathrm{ngd}} \subset \mathcal{M}^e$ with *economic meaning*
  - Choose $\mathcal{Q}^{\mathrm{ngd}}$ so that "too good deals" are excluded for any market extension $\big(S_t, E_t^Q[X]\big)$ for any contingent claim X when $Q \in \mathcal{Q}^{\mathrm{ngd}}$.
  - Define upper and lower good-deal valuation bounds

$$\pi_t^u(X) := \operatorname*{esssup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X], \quad \pi_t^l(X) := -\pi^u(-X).$$

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- **Hedging Strategy:**minimizes a-priori coherent risk measure ρ over all strategies ϕ for optimal risk sharing with the market:

$$\pi_t^u(X) = \rho_t\left(X - \int_t^T \bar{\phi}_s dS_s\right) = \operatorname*{essinf}_{\phi \in \Phi} \rho_t\left(X - \int_t^T \phi_s dS_s\right) \quad \forall t \in [0, T].$$

- Solution for dominated uncertainty, solely about drift, via (standard) BSDE, g-expectations: [B., Kentia, MMOR 2017]

Aspect II: non-dominated vs dominated uncertainty

- For super-replication price v only non-dominated uncertainty matters, but not drift (or market price of risk) uncertainty:
- Example:
  Black-Scholes-type model $dS_t = S_t\sigma_t(\xi_t dt + dW_t)$
- Black-Scholes pricing pde :

$$\sigma^2 s^2 v_{ss} + v_t = 0$$

Aspect II: non-dominated vs dominated uncertainty

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- Example:
  Black-Scholes-type model $dS_t = S_t\sigma_t(\xi_t dt + dW_t)$
  with **(non-dominated) volatility uncertainty**

$$0 < \underline{\sigma}^2 \le \sigma_t(\omega)^2 \le \overline{\sigma}^2$$

- Black-Scholes-Barenblatt pricing pde (fully non-linear):

$$(\overline{\sigma}^2 1_{\{v_{ss}\ge 0\}} + \underline{\sigma}^2 1_{\{v_{ss}<0\}})s^2 v_{ss} + v_t = 0$$

- BSDE-context: more general path-dependent claims, non-Makovian setup $\rightsquigarrow$ 2nd order BSDE or G-expectation
- For any given $\sigma > 0$ model is complete here. But what to do in **generically incomplete** models if superreplication is too expensive?

Outline

1. Formulation in the absence of ambiguity
   - No-good-deal restriction
   - Valuation and hedging

2. Valuation and hedging under ambiguity about drift and volatility
   - Framework for combined drift and volatility uncertainty
   - Robust valuation and hedging via 2BSDEs

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Financial market and no-good-deal pricing measures

- Fixed $(\Omega, \mathcal{F}, \mathbb{F}, P)$, W n-dimensional P-Wiener process, $\mathbb{F} = \overline{\mathbb{F}^W}^P$.
- Discounted $(r = 0)$ prices of d traded stocks $S = (S^i)_{i=1}^d$:

$$dS_t = \mathrm{diag}(S_t)\sigma_t(\xi_t dt + dW_t), \quad t \in [0, T],$$

  market price of risk $\xi \in \mathrm{Im}\,\sigma^{\mathrm{tr}}$, volatility σ of maximal rank $d < n$.
- Choose $\mathcal{Q}^{\mathrm{ngd}}(P)$ consisting of $Q \in \mathcal{M}^e(P)$ with $Z := \frac{dQ}{dP}$ satisfying

$$E^P_\tau\left[-\log\frac{Z_\sigma}{Z_\tau}\right] \le E^P_\tau\left[\frac{1}{2}\int_\tau^\sigma h_s^2 ds\right], \quad \text{for all } \tau \le \sigma \le T,$$

  for $h > 0 \rightsquigarrow$ bounds *cond. reverse rel. entropy* of Q wrt. P.

Economic meaning of no-good-deals

- This yields a bound on Girsanov kernels of pricing measures:

$$\mathcal{Q}^{\mathrm{ngd}}(P) = \Big\{ Q \in \mathcal{M}^e(P) : dQ/dP = \mathcal{E}\left(\lambda \cdot W\right), \text{ and } |\lambda| \leq h \Big\}.$$

which equivalently corresponds to imposing bounds on

- **instantaneous Sharpe ratios:** $SR_t(N) := \frac{\text{mean excess return(t)}}{\text{standard deviation (t)}} \leq h_t$, i.e. $SR_t(N) = \mu_t^N / \sigma_t^N$ for return $dN_t/N_t = \mu_t^N dt + \sigma_t^N dW_t$,
- **optimal** P**-expected growth rate** of returns:

$$E_\tau^P\left[\log \frac{N_\sigma}{N_\tau}\right] \leq E_\tau^P\left[-\log \frac{Z_\sigma^Q}{Z_\tau^Q}\right] \leq E_\tau^P\left[\frac{1}{2}\int_\tau^\sigma h_s^2 ds\right], \ \forall \tau \leq \sigma,$$

for all $Q \in \mathcal{Q}^{\mathrm{ngd}}$ and Q-local martingales $N > 0$, i.e. in any financial market extension by additional derivatives.

Admissible trading strategies

- Trading strategy φ as amount invested into stocks $\rightsquigarrow$ wealth process

$$dV_t := \varphi_t^{\mathrm{tr}} \frac{dS_t}{S_t} = \varphi_t^{\mathrm{tr}} \sigma_t (\xi_t dt + dW_t)$$

- Convenient to re-parameterize strategies as $\phi := \sigma^{\mathrm{tr}} \varphi \in \mathrm{Im}\, \sigma^{\mathrm{tr}}$:

$$dV_t = \phi_t^{\mathrm{tr}} (\xi_t dt + dW_t) =: \phi_t^{\mathrm{tr}} d\widehat{W}_t.$$

- Denote the set of P-admissible trading strategies by $\Phi(P)$.

Good-deal hedging: minimize coherent risk

- a-priori dynamic coherent time-consistent risk measure ρ^P:

$$\rho_t^P(X) := \operatorname*{esssup}_{Q\in\mathcal{P}^{\mathrm{ngd}}(P)} E_t^Q[X], \quad t\in[0,T],$$

for $\mathcal{P}^{\mathrm{ngd}}(P) := \Big\{Q\sim P \ : dQ/dP = \mathcal{E}\left(\lambda\cdot W\right), \text{ with } |\lambda|\le h\Big\}$.

- $\mathcal{Q}^{\mathrm{ngd}}(P) = \mathcal{P}^{\mathrm{ngd}}(P)\cap\mathcal{M}^e(P)$.
- Hedging objective: Find strategy $\bar\phi = \bar\phi^P \in \Phi(P)$ s.t.

$$\pi_t^{u,P}(X) = \rho_t^P\Big(X - \int_t^T \bar\phi_s^{\mathrm{tr}} d\widehat{W}_s\Big) = \operatorname*{essinf}_{\phi\in\Phi(P)} \rho_t^P\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\Big).$$

- Good deal bounds π^u as market-based risk measures from minimizing coherent risk ρ, hedging as optimal 'risk-sharing' with the market, valuation & hedging 'to aceptability' [cf. Barrieu et al, Madan et al...]

Hedging error and its supermartingale characterization

- For strategy $\phi \in \Phi(P)$, its tracking (or hedging) error is

$$L_t^{\phi}(X) := \underbrace{\pi_t^{u,P}(X) - \pi_0^{u,P}(X)}_{\text{variation of capital requirement}} - \underbrace{(V_t^{\phi} - V_0^{\phi}),}_{\text{P\&L from dyn.trading by } \phi} \qquad t \in [0, T].$$

- Tracking error $L^{\bar{\phi}}$ of the good-deal hedging strategy $\bar{\phi}^P$ is Q-supermartingale for any $Q \in \mathcal{P}^{\mathrm{ngd}}(P)$.
  (sufficient and necessary condition)

- $\rightsquigarrow$ Hedging strategy $\bar{\phi}^P$ is at least mean-self-financing under any $Q \in \mathcal{P}^{\mathrm{ngd}}(P)$.

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Motivation

- **Problem:**
  market prices of risk ξ (drift) and volatilities σ are not known:
  we do not know the objective real world measure P precisely...
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- Instead of single reference measure P, consider set $\mathcal{R}$ of plausible reference priors capturing ambiguity about both drift and volatility.

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- **Problem:**
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  we do not know the objective real world measure P precisely...
- There is ambiguity $\rightsquigarrow$ Knightian model uncertainty.
- Instead of single reference measure P, consider set $\mathcal{R}$ of plausible reference priors capturing ambiguity about both drift and volatility.
- **Aim:** Robust good-deal valuation and hedging w.r.t. uncertainty about ξ and σ.

Local martingale measures on canonical space

- $\Omega := C_0([0,T],\mathbb{R}^n)$, B coordinate process, P_0 Wiener measure, $\mathbb{F} := (\mathcal{F}_t)_{t\leq T}$ filtration generated by B, $\mathbb{F}^+$ right limit of $\mathbb{F}$.
- $\overline{\mathcal{P}}_S$: collection of all local martingale measures P_α for B with

$$P_\alpha := P_0 \circ (X^\alpha)^{-1}, \text{ where } X^\alpha_t := \int_0^t \alpha_s^{\frac{1}{2}}\, dB_s, \quad P_0\text{-a.s. } t \in [0,T],$$

  with positive-definite-valued α, $\mathbb{F}$-prog. meas., $\int_0^T |\alpha_s|\, ds < \infty$, P_0-a.s..
- E.g. Black-Scholes model with different volatilities.
- Karandikar '95: The stochastic integral $\int_0^\cdot B_s dB_s^{\mathrm{tr}}$ can be defined ω-wise such that it coincides with Itô integral P-a.s. for all $P \in \overline{\mathcal{P}}_S$.
- Then quadratic variation $\langle B\rangle_t$ and density $\widehat{a}_t := \frac{d\langle B\rangle_t}{dt}$ are also defined ω-wise.

Non-dominated measures - quasi-sure analysis

- **Note:** The probability measures in $\overline{\mathcal{P}}_S$ may be mutually singular.
- For fixed $\underline{a}, \overline{a} \in \mathbb{S}_n^{>0}$ (set of positive-definite matrices), consider subclass $\mathcal{P} \subset \overline{\mathcal{P}}_S$ defined by

$$\mathcal{P} := \left\{ P \in \overline{\mathcal{P}}_S : \; \underline{a} \leq \widehat{a} \leq \overline{a}, \; P \otimes dt\text{-a.s.} \right\}.$$

- $\underline{a} \leq \widehat{a} \leq \overline{a}$ $\quad \rightsquigarrow$ confidence region for volatilities scenarios.
- **Def**: A property holds $\mathcal{Q}$-quasi-everywhere (shortly $\mathcal{Q}$-q.e.) for family $\mathcal{Q}$ of measures if it holds outside a set which is Q-negligible $\forall Q \in \mathcal{Q}$.

Financial market with uncertainty

Risky asset prices $S = (S^i)_{i=1}^d$ modeled as

$$dS_t = \mathrm{diag}(S_t)\,(b dt + \sigma dB_t),\ \mathcal{P}\text{-q.s.}, \quad S_0 \in (0,\infty)^d,$$

with $\sigma \in \mathbb{R}^{d\times n}$ ($d < n$) and $\sigma \widehat{a}^{\frac{1}{2}}$ $\mathcal{P} \otimes dt$-q.e. of maximal rank d.

- $W^P := \int_0^\cdot \widehat{a}_s^{-\frac{1}{2}} dB_s$ is a P-Brownian motion, for each $P \in \mathcal{P}$.
- **Volatility uncertainty:** $\sigma \widehat{a}^{\frac{1}{2}}$ plays the role of volatility matrix for stock prices S under each $P \in \mathcal{P}$, since $dB_t = \widehat{a}_t^{\frac{1}{2}} dW_t^P$.
- $\sigma \widehat{a}^{\frac{1}{2}}$ of maximal rank $d < n \rightsquigarrow$ incomplete market under each $P \in \mathcal{P}$.
- (Minimal) market prices of risk in each model $P \in \mathcal{P}$ given by

$$\widehat{\xi}_t := \widehat{a}_t^{\frac{1}{2}} \sigma^{\mathrm{tr}} (\sigma \widehat{a}_t \sigma^{\mathrm{tr}})^{-1} b,\ t \le T, \quad P\text{-a.s.}.$$

Plausible reference measures

- Consider candidate market prices of risk and volatilities

$$\xi_t^{P,\theta} = \widehat{\xi}_t + \theta_t \quad \text{and} \quad \sigma_t^{P,\theta} = \sigma \widehat{a}_t^{\frac{1}{2}},$$

for θ_t in some ellipsoidal confidence region $\Theta_t \subset \mathrm{Im}\left(\sigma_t \widehat{a}_t^{\frac{1}{2}}\right)^{\mathrm{tr}}$, $t \le T$.

- Corresponding set of reference priors for drift & vol. uncertainty is

$$\mathcal{R} := \left\{ Q = Q^{P,\theta} \;\middle|\; Q \sim P,\ \frac{dQ}{dP} = {}^{(P)}\mathcal{E}(\theta \cdot W^P),\ \text{for some } P \in \mathcal{P},\ \theta_t \in \Theta_t\ \forall t \right\}.$$

- For each $P \in \mathcal{P}$ holds $\mathcal{M}^e(Q^{P,\theta}) = \mathcal{M}^e(P)$ for any $\theta \in \Theta$.
- Set of no-good-deal pricing measures for each prior $Q^{P,\theta} \in \mathcal{R}$:

$$\mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta}) = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; \frac{dQ}{dP} = {}^{(P)}\mathcal{E}(\lambda \cdot W^P),\ |\lambda + \theta| \le h \right\}.$$

Uncertainty: Worst-case bounds and a-priori risk measure

- For each $Q^{P,\theta} \in \mathcal{R}$, consider the sets $\mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta})$ and $\mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})$.
- For model $Q^{P,\theta}$, good-deal bound and risk measure are

$$\pi_t^{u,P,\theta}(X) = \operatorname*{esssup^{P}}_{Q \in \mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta})} E_t^Q[X] \quad , \quad \rho_t^{P,\theta}(X) = \operatorname*{esssup^{P}}_{Q \in \mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})} E_t^Q[X], \ P\text{-a.s..}$$

- Robust worst-case good-deal bound under uncertainty:

$$\pi_t^u(X) := \operatorname*{esssup^{P}}_{P' \in \mathcal{P}(t+,P)} \ \operatorname*{esssup^{P}}_{\theta \in \Theta} \pi_t^{u,P',\theta}(X), \ P\text{-a.s.,} \quad \forall P \in \mathcal{P}$$

where $\mathcal{P}(t+,P) = \{P' \in \mathcal{P} : P' = P \,\mathrm{on}\, \mathcal{F}_t^+\}$.

- a-priori risk measure to be minimized by dynamic hedging:

$$\rho_t(X) := \operatorname*{esssup^{P}}_{P' \in \mathcal{P}(t+,P)} \ \operatorname*{esssup^{P}}_{\theta \in \Theta} \rho_t^{P',\theta}(X), \ P\text{-a.s.,} \quad \forall P \in \mathcal{P}.$$

2BSDE formulation and wellposedness

For a measurable generator $F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n^{>0} \to \mathbb{R}$, Lipschitz in (y, z), denote $\widehat{F}_t(y, z) := F_t(B_{\cdot\wedge t}, y, z, \widehat{a}_t)$.

- For $\mathcal{F}_T$-measurable X, define second-order BSDE (2BSDE)

$$Y_t = X - \int_t^T \widehat{F}_s(Y_s, \widehat{a}_s^{\frac{1}{2}} Z_s)ds - \int_t^T Z_s dB_s + K_T^P - K_t^P, \ \mathcal{P}\text{-q.s.}$$

- Solution triple $(Y, Z, (K^P)_{P\in\mathcal{P}})$, satisfies minimum condition

$$K_t^P = \underset{P'\in\mathcal{P}(t+,P)}{\operatorname{essinf}^P} E_t^{P'}[K_T^{P'}], \ P\text{-a.s.}, \ t \in [0, T], \ \forall P \in \mathcal{P}.$$

- Wellposedness under suitable measurability and integrability properties on X and F (Possamai/Tan/Zhou)

Good-deal valuation via 2BSDEs

- For each $a \in \mathbb{S}_n^{>0}$ and $t \in [0, T]$, consider the orthogonal projections

$$\Pi_t^a : \mathbb{R}^n \to \mathrm{Im}\left(\sigma_t a^{1/2}\right)^{\mathrm{tr}} \quad \text{and} \quad \Pi_t^{\perp,a} : \mathbb{R}^n \to \mathrm{Ker}\left(\sigma_t\, a^{1/2}\right).$$

- Let F be the generator function defined by

$$F_t(z,a) := \inf_{\theta \in \Theta} \left(\widehat{\xi}_t^{\,\mathrm{tr}} \Pi_t^a(z) - \sqrt{h_t^2 - |\widehat{\xi}_t + \theta_t|^2}\,\left|\Pi_t^{\perp,a}(z)\right| \right)$$

- For suitable X, there exists a unique (in a suitable space) solution $(Y, Z, (K^P)_{P \in \mathcal{P}})$ to 2BSDE with parameters (F, X).
- Worst-case good-deal bound process is given by

$$\pi_t^u(X) = Y_t,\ P\text{-a.s.},\ \forall P \in \mathcal{P}$$

.

Good-deal hedging via 2BSDEs

- Again trading strategy as amount of wealth invested in S
  $\rightsquigarrow$ wealth process

$$V^{\phi} = V_0 + \int_0^{\cdot} \phi_s^{\mathrm{tr}} \big(\widehat{a}_s^{\frac{1}{2}} \xi_s ds + dB_s \big), \quad \text{with } \phi \in \operatorname{Im} \sigma^{\mathrm{tr}}.$$

- Set Φ of $\mathcal{R}$-admissible strategies $\phi \in \Phi$ such that trading gains $\left\{ {}^{(P)}\!\int_0^{\cdot} \phi_s^{\mathrm{tr}} dB_s,\ P \in \mathcal{P} \right\}$ aggregate into single process $\int_0^{\cdot} \phi_s^{\mathrm{tr}} dB_s$.
- Aggregation condition is not needed under additional set-theoretical axioms. Otherwise it holds e.g. if ϕ is càdlàg.
- Hedging problem under drift and vol. uncertainty: Find $\bar{\phi} \in \Phi$ s.t.

$$\pi_t^u(X) = \rho_t\Big(X - \big(V_T^{\bar{\phi}} - V_t^{\bar{\phi}} \big) \Big) = \operatorname*{essinf}_{\phi \in \Phi} \ \rho_t\Big(X - \big(V_T^{\phi} - V_t^{\phi} \big) \Big).$$

Good-deal hedging via 2BSDEs (cont.)

Denote $\widehat{\Pi}_t := \Pi_{\mathrm{Im}\,(\sigma_t \widehat{a}^{1/2})^{\mathrm{tr}}}$ and $\widehat{\Pi}_t^{\perp} := \Pi_{\mathrm{Ker}\,(\sigma_t \widehat{a}^{1/2})}$, and recall $(Y, Z, (K^P)_{P\in\mathcal{P}})$ solution to the 2BSDE for $\pi^u_\cdot(X)$.

- Given trading gains $\{^{(P)}Z \cdot B,\ P \in \mathcal{P}\}$ aggregate, then for some worst-case $\bar{\theta} \in \Theta$ the hedging strategy is given by

$$\widehat{a}_t^{1/2}\bar{\phi}_t(X) = \underbrace{\widehat{\Pi}_t\big(\widehat{a}_t^{1/2} Z_t\big)}_{\text{Non-speculative component}} + \underbrace{\frac{\big|\widehat{\Pi}_t^{\perp}\big(\widehat{a}_t^{1/2} Z_t\big)\big|}{\sqrt{h_t^2 - \big|\xi_t + \bar{\theta}_t\big|^2}}\big(\xi_t + \bar{\theta}_t\big)}_{\text{Speculative component}}.$$

- Robust tracking error $L^{\bar{\phi}} = \pi^u_\cdot(X) - \pi^u_0(X) - (V^{\bar{\phi}}_T - V^{\bar{\phi}}_\cdot)$ is a Q-supermartingale for any $Q \in \mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})$ for all $P \in \mathcal{P}, \theta \in \Theta$

  $\ldots \rightsquigarrow$ at least mean-self.fin. wrt. drift & volatility uncert.

a particular (rare) example with an explicit solution:

- Market with two Black-Scholes-type assets ($d = 1,\ n = 2$):

$$dS_t = S_t \sigma^S dB_t^1, \quad dH_t = H_t\big(\gamma dt + \beta(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2)\big),\ \mathcal{P}\text{-q.s.},$$

  for $\rho \in [-1, 1]$, $\widehat{a} = (\widehat{a}^{ij})_{i,j=1,2} \in [\underline{a}, \overline{a}]$,
  no drift uncertainty, market price of risk $\xi = 0$.

- Put option $X = (\mathcal{K} - H_T)^+$ **on non-traded asset** H, and $h \in [0, \infty)$.
- Valuation at maximum vol. level $\overline{a}$, for worst-case model $P_{\overline{a}} \in \mathcal{P}$:

$$\pi_t^u(X) = \pi_t^{u,P_{\overline{a}}}(X) = C * \text{Black-Scholes-Put-price}\Big(\text{spot } H_t, \text{ strike } \frac{\mathcal{K}}{C}, \text{ vol. } \bar{\beta}\Big),$$

  for some $C, \bar{\beta} \in (0, \infty)$.

- Hedging strategy: $\bar{\phi}_t(X) = L_t\, H_t\Big(\rho + \frac{\widehat{a}_t^{12}}{\widehat{a}_t^{11}}\sqrt{1-\rho^2}, 0\Big)^{\mathrm{tr}}$ for some $L_t < 0$.
- Incompleteness ($|\rho| \neq 1$) $\rightsquigarrow$ Good-deal hedging $\neq$ Super-replication!

Summary

Valuation and hedging under combined drift and volatility uncertainty.

1. Good-deal approach yields less expensive valuations and less extreme hedges than (quasi-sure) superreplication.
2. Hedging strategies are at-least-mean-self-financing, uniformly over all a-priori valuation measures wrt. all (uncertain) reference priors.
3. Valuations and Hedges are characterized by the solution to a 2nd-order BSDEs, for general measurable claims (no continuity conds), by building on wellposedness from D.Possamai,X.Tan,C.Zhou for 2BSDE, where generator needs not to be convex or continuous.
4. Combined uncertainty about drift and volatility is complicated but it matters !

t h a n k   y o u

1 B., Klebert Kentia: Good deal hedging and valuation under combined uncertainty about drift and volatility, to app. in *Probability Uncertainty and Quantitative Risk*, https://ssrn.com/abstract=2951742

2 B., Klebert Kentia: Hedging under generalized good-deal bounds and model uncertainty, *Math Meth Oper Res*, 2017 (dominated uncertainty only), http://dx.doi.org/10.1007/s00186-017-0588-y

(preprints also on arXiv)