Crashes and Bubbles: a heterogeneous agent model with transaction costs and learning

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Big Picture

- Suppose that we must make a decision.
- Classical theory suggests some abstract techniques that we could use to guide our choices.
- If we adopted such methods, generally, we would need to specify our beliefs about the factors affecting the decision's outcome:

  What are these factors?

  What are their characteristics?
- We would determine how to act, based on the assumption that our replies were *completely correct.*

Big Picture

What if we were wrong?

It might not matter.

It might be a disaster!

Motivation

The Flash Crash

During the Flash Crash on May 6, 2010, "over 20,000 trades across more than 300 securities were executed at prices more than 60% away from their values just moments before. Moreover, many of these *trades were executed at prices of a penny or less, or as high as $100,000,* before prices of those securities returned to their 'pre-crash' levels."

(CFTC & SEC, 2010)

Motivation
Mini Flash Crashes (Frequency)

- While the scale of the Flash Crash was unusual, *mini flash crashes* occur quite often.
- Anecdotal evidence from traders suggests that such events *happen over a dozen times each day* (Farrell, 2013).
- A rigorous empirical analysis uncovered "*18,520 crashes and spikes with durations less than 1,500 ms*"' in stock prices from 2006 through 2011 (Johnson et al., 2013).

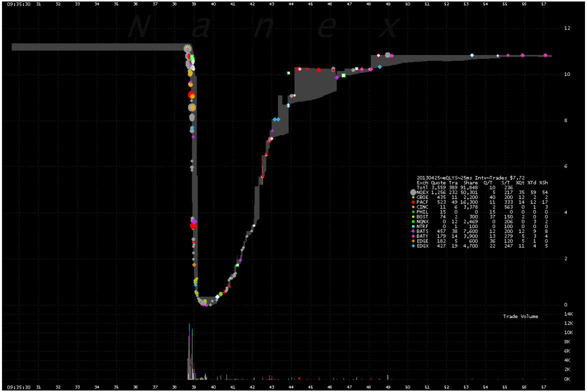
Motivation
Mini Flash Crashes (Definition)

- *Popular Definition:* An event in which the price of some security changes at least 0.8% and ticks at least ten times consecutively in one direction.

Motivation

Mini Flash Crashes (Definition)

- Swings may not be so mild: The SEC classified jumps in QYLS from $10 to $0.0001 and back during a 300ms period on April 25, 2013 as a *mini* flash crash!



Motivation

Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Human error/improper risk management?
- *Example:* In 2005, Mizuho Securities attempted to sell \$2.9 billion worth of shares in a firm worth only \$90 million, causing an \$875 price/share drop in 30m.
- *Example:* Merrill Lynch's systems allowed the submission of order sizes that were 60 times a security's daily volume, leading to over 15 mini flash crashes.

Motivation
Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Fundamental value perception?
- *Example:* A false tweet sent by the AP's (hacked) Twitter account erased $136 billion from the S&P 500 Index in only two minutes on April 23, 2013.

Motivation

Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Regulations?
- *Idea:* Most liquidity providers have no formal obligations.
- *Idea:* Market-stabilizing orders risk nullification.
- *Idea:* Order routing guidelines may adversely affect execution.

Motivation

Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Market fragmentation?
- *Idea:* Certain order types such as intermarket market sweep orders can force execution at a particular exchange.
- *Idea:* If liquidity is comparably low on the designated exchange, trades will be filled at extremely adverse prices even when more favorable prices are available on other exchanges (Chakravarty et al., 2012).

Motivation

Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Perceived order flow toxicity?
- *Idea:* Conditions including unusual order book imbalance and trade intensity might foreshadow market maker adverse selection, drying up liquidity (O'Hara et al., 2012).
- *Example:* The CFTC charged Navinder Sarao with unlawfully creating an "order book imbalance that contributed to market conditions that led to the Flash Crash" (CFTC, 2015).

Motivation

Mini Flash Crashes (Possible Causes)

- *Vanishing Liquidity Theory:* Thin liquidity coupled with synchronized single-direction trading cause mini flash crashes!

Why?

- *Possible Reason:* Positive feedback loops?
- *Idea:* "Crowds of agents frequently converge on the same strategy and hence simultaneously flood the market with the same type of order, thereby generating the frequent extreme price-change events" (Johnson et al., 2013).
- *Example:* "95% of the [peak Flash Crash] trading was due to endogenous triggering effects" (Sornette et al., 2012).

Motivation

Mini Flash Crashes (Preventive Measures)

- *Limit Up-Limit Down Rule:* The SEC temporarily suspends trading in individual securities whose prices escape certain upper and lower bounds in specified short periods.

Motivation

Mini Flash Crashes (Preventive Measures)

- *Market-Wide Circuit Breakers:* The SEC can also temporarily halt all trading when the S&P 500 Index declines sufficiently in a single trading day.
- *Example:* The SEC will impose a *Level 1* suspension of market-wide trading for 15 minutes, if the single-day decline in the S&P 500 Index hits 7% before 3:25 PM.
- *Example:* If the S&P 500 Index drops 7% at 3:25 PM or later, trading is halted for the rest of the day.

Our Approach

- *Step 1:* Describe a collection of heterogenous agents, each of whom has some model for a risky asset's price and trades to maximize an objective function subject to certain constraints.

- *Step 2:* Solve the problem faced by a single generic agent (each agent will only indirectly account for the others, i.e., we do *not* study Nash equilibria).

- *Step 3:* Specify the *actual* price dynamics (no agent has complete information about the others, and in particular, every agent's model will be *slightly wrong*).

- *Step 4:* Analyze the system, assuming knowledge of all models and parameters (call this the *DG perspective*).

Our Results

- low agent confidence in model drift parameters;
- agent underestimation of temporary market impact;
- large agent populations;
- long trading horizons.

Our Agents

- We consider a population of N agents: Agents $1 - N$.
- There is a special nonnegative parameter ν_j^2 associated to Agent j.
- *Definition:* If $\nu_j^2 > 0$, then we call Agent j an *uncertain* agent.
- *Definition:* If $\nu_j^2 = 0$, then we call Agent j a *certain* agent.

Our Agents

- There are $K \in \{0, \dots, N\}$ uncertain agents, namely, Agents 1 through K.
- All agents *attempt to* trade in a single risky asset over a time horizon $[0, T]$.
- Our agents trade continuously by optimally selecting a trading rate from a particular class of admissible strategies.
- All trades submitted at time t are executed immediately at the price S_t^{exc}.

Our Agents
Models

- At each time t, every agent observes the correct value of S_t^{exc}.
- No agent knows the true dynamics of the stochastic process S^{exc}, though.
- Instead, prior to $t = 0$, Agent j has developed a model S_{j,θ_j}^{exc} for S^{exc}.

Our Agents
Models

- *Definition:* The $\mathscr{F}_{j,t}$-adapted process S_j^{unf} given by

$$S_{j,t}^{unf} \triangleq S_{j,0} + \beta_j t + W_{j,t} \qquad (1)$$

is Agent j's estimate of the *unaffected* or *fundamental* price of the risky asset at time t.

- *Definition:* W_j is an $\mathscr{F}_{j,t}$-Wiener process under P_j.

- *Definition:* β_j is an $\mathscr{F}_{j,0}$-measurable random variable, which is independent of W_j and normally distributed with mean μ_j and variance ν_j^2 under P_j.

- *Definition:* $S_{j,0}$ is a deterministic constant known to Agent j.

Our Agents
Models

- Let $\theta_{j,t}$ denote Agent j's trading rate at time t
- *Agent's model:* The $\mathscr{F}_{j,t}$-adapted process S^{exc}_{j,θ_j} given by

$$S^{exc}_{j,\theta_j,t} \triangleq S^{unf}_{j,t} + \eta_{j,per} \int_0^t \theta_{j,s}\, ds + \frac{1}{2}\eta_{j,tem}\theta_{j,t} \tag{2}$$

  is Agent j's model of the *execution price* at time t.
- $\eta_{j,per}$ and $\eta_{j,tem}$ are deterministic positive constants Agent j uses to model her price impact. (Almgren-Chriss (2001) price impact model.)

Our Agents
Admissible Strategies

- *Definition:* Let $\mathscr{A}_j$ be the space of $\mathscr{F}^{unf}_{j,t}$-adapted processes θ_j such that $\theta_{j,\cdot}(\omega)$ is continuous on $[0,T]$ for P_j-almost every $\omega \in \Omega_j$,
$$E^{P_j}\left[\int_0^T \theta_{j,t}^2\,dt\right] < \infty \quad P_j - \text{a.s.}, \tag{3}$$
and
$$x_j + \int_0^T \theta_{j,t}\,dt = 0 \quad P_j - \text{a.s.}. \tag{4}$$

- *Definition:* For any $\theta_j \in \mathscr{A}_j$, we define the process $X_j^{\theta_j}$ by
$$X_{j,t}^{\theta_j} = x_j + \int_0^t \theta_{j,s}\,ds. \tag{5}$$

Our Agents
Objective Functions

- *Definition:* Agent j's objective is to maximize

$$E^{P_j}\left[-\int_0^T \theta_{j,t}\, S^{exc}_{j,\theta_j,t}\, dt - \frac{\kappa_j}{2}\int_0^T \left(X^{\theta_j}_{j,t}\right)^2\, dt\right] \qquad (6)$$

over $\theta_j \in \mathscr{A}_j$.

Objectives

$$\theta_j \longmapsto E^{P_j}\left[-\int_0^T \theta_{j,t}\, S^{exc}_{j,\theta_j,t}\, dt - \frac{\kappa_j}{2}\int_0^T \left(X^{\theta_j}_{j,t}\right)^2 dt\right] \tag{6}$$

- *Note:* Using an innovation process and integration by parts, one can show that maximizing (6) is equivalent to maximizing

$$\theta_j \longmapsto E^{P_j}\left[\int_0^T X^{\theta_j}_{j,t}\, E^{P_j}\left[\beta_j \middle| \mathscr{F}^{unf}_{j,t}\right] dt - \frac{\kappa_j}{2}\int_0^T \left(X^{\theta_j}_{j,t}\right)^2 dt - \frac{1}{2}\eta_{j,tem}\int_0^T \theta^2_{j,t}\, dt\right]. \tag{7}$$

- *Note:* Variants of (7) have appeared previously in the literature (Garleanu, Pedersen, 2013).

Our Agents
Optimal Strategies

- Agent j's optimization problem has a *unique* solution $\theta_j^\star \in \mathscr{A}_j$.

- When $\omega \in \Omega_j$ is chosen such that $W_{j,\cdot}(\omega)$ is continuous, $X_j^{\theta_j^\star}(\omega)$ satisfies the linear ODE: $X_{j,0}^{\theta_j^\star}(\omega) = x_j$,

$$\theta_{j,t}^\star(\omega) = -\sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth\left(\tau_j(t)\right) X_{j,t}^{\theta_j^\star}(\omega) + \frac{\tanh\left(\tau_j(t)/2\right)\left[\mu_j + \nu_j^2\left(S_{j,t}^{unf}(\omega) - S_{j,0}\right)\right]}{\sqrt{\eta_{j,tem}\kappa_j}\left(1+\nu_j^2 t\right)}, \tag{8}$$

where

$$\tau_j(t) \triangleq \sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\left(T-t\right).$$

Our Agents
Optimal Strategies

- In particular, if Agent j is *certain*,

$$\theta^{\star}_{j,t} = -\sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth\left(\tau_j\left(t\right)\right)X^{\theta^{\star}_j}_{j,t} + \frac{\mu_j \tanh\left(\tau_j\left(t\right)/2\right)}{\sqrt{\eta_{j,tem}\kappa_j}},$$
$$X^{\theta^{\star}_j}_{j,0} = x_j. \tag{9}$$

- *Remark:* Can be converted to a tracking problem and then one can use Bank, Soner, Voss (2017).

Execution Price

- While each agent observes the same realized path of this process, in general, no agent knows the correct dynamics.
- An agent's trading decisions are entirely determined by his beliefs, preferences, and observations of a single realized path of S^{exc}.

Execution Price

- *Definition:* The true execution price S^{exc} under $\tilde{P}$ is the $\tilde{\mathscr{F}}_t$-adapted process

$$S_t^{exc} = S_0 + \tilde{\beta}t + \sum_{i=1}^{N} \tilde{\eta}_{i,per}\left(X_{i,t}^{\theta_i^\star} - x_i\right) + \frac{1}{2}\sum_{i=1}^{N} \tilde{\eta}_{i,tem}\theta_{i,t}^\star + \tilde{W}_t \tag{10}$$

- We have the following deterministic real constants:

$$\tilde{\beta}, \quad S_0, \quad \tilde{\eta}_{1,per}, \dots, \tilde{\eta}_{N,per}, \quad \text{and} \quad \tilde{\eta}_{1,tem}, \dots, \tilde{\eta}_{N,tem}$$

- Á la Carlin, Lobo, Visvanathan (2007), but without games.

Execution Price

- *Remark:* Comparing our descriptions of S^{exc}_{j,θ_j} in (2) and S^{exc} in (10), we see that Agent j proxies each term in (10) as follows:

$$
\begin{aligned}
\eta_{j,per}\left(X^{\theta^\star_j}_{j,t} - x_j\right) \quad &\rightsquigarrow \quad \tilde{\eta}_{j,per}\left(X^{\theta^\star_j}_{j,t} - x_j\right) \\
\frac{1}{2}\eta_{j,tem}\theta^\star_{j,t} \quad &\rightsquigarrow \quad \frac{1}{2}\tilde{\eta}_{j,tem}\theta^\star_{j,t} \\
S_{j,0} + \beta_j t + W_{j,t} \quad &\rightsquigarrow \quad S_0 + \tilde{\beta} t + \sum_{i \neq j} \tilde{\eta}_{i,per}\left(X^{\theta^\star_i}_{i,t} - x_i\right) \\
&\qquad + \frac{1}{2}\sum_{i \neq j} \tilde{\eta}_{i,tem}\theta^\star_{i,t} + \tilde{W}_t.
\end{aligned}
$$

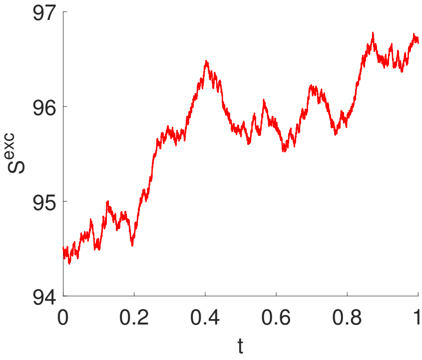
Key Question

When the agents trade according to the strategies they believe to be optimal and S^{exc} has the described dynamics, what happens?

Case 1
No Mini-Flash Crashes

- There are broad sufficient conditions on our deterministic parameters such that S^{exc}, the $X_j^{\theta_j^\star}$'s and the $\theta_j^\star$'s are all uniquely defined and continuous on $[0, T]$ and

$$\lim_{t \uparrow T} X_t^{\theta^\star} = 0 \quad \tilde{P} - a.s..$$



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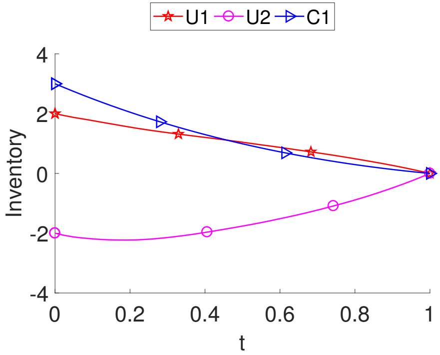

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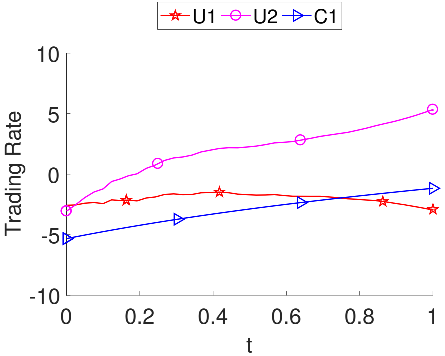
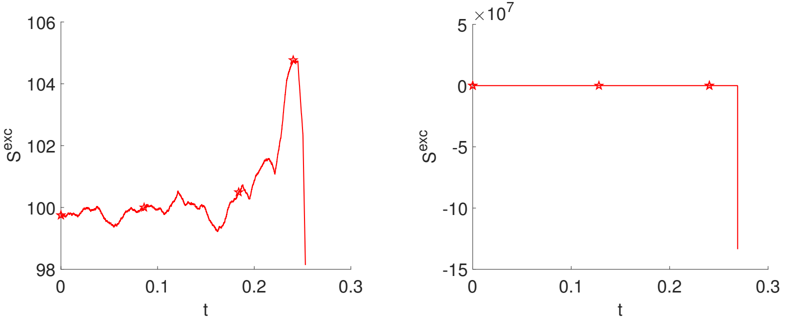


Case 2

High Trading Volume Mini-flash Crashes

- There are broad sufficient conditions on our deterministic parameters such that S^{exc}, the $X_j^{u,\theta_j^\star}$'s, and the $\theta_j^{u,\star}$'s all explode in the same random direction as $t \uparrow t_e$ $\tilde{P}$-a.s.



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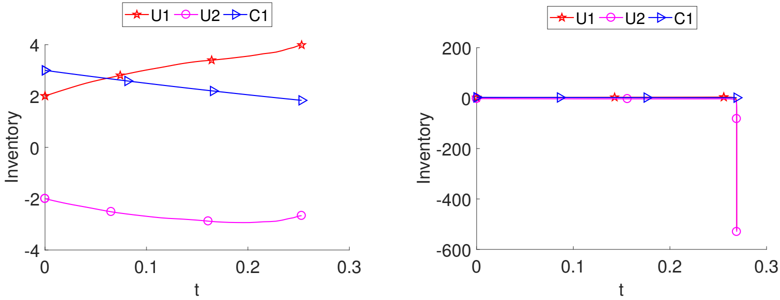

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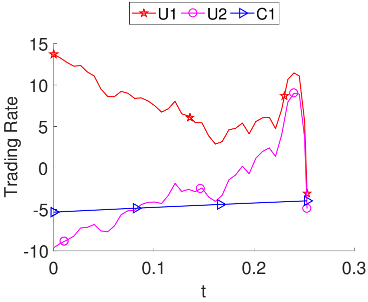
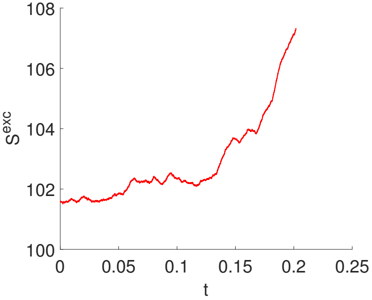
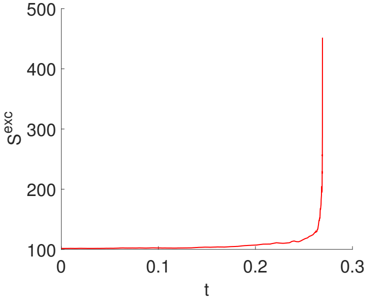


Case 3

Low Trading Volume Mini-flash Crashes

- There are broad sufficient conditions on our deterministic parameters such that all coordinates of $X^{\theta^\star}$ have a finite limit but S^{exc} and the $\theta_j^{u,\star}$'s all explode in the same random direction as $t \uparrow t_e$ $\tilde{P}$-a.s.





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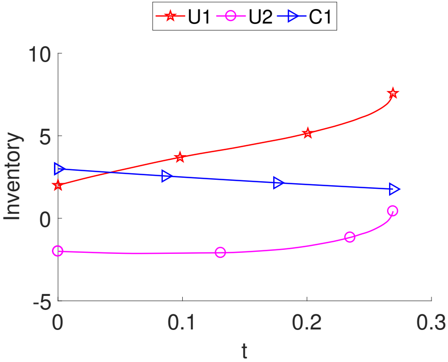

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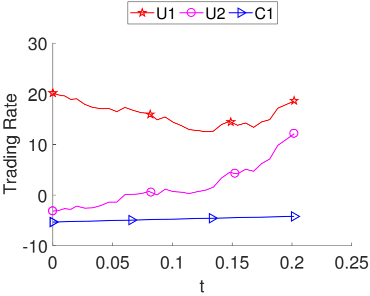

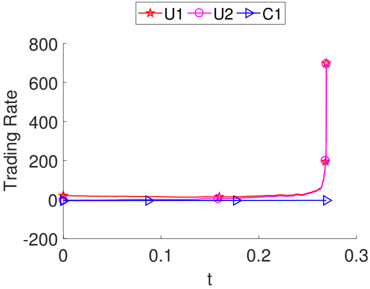

DG System

- In what follows, we adopt the *DG perspective*, that is, we assume knowledge of all models and parameters.
- Suppose that $\omega \in \hat{\Omega}$ is *actually* realized.
- *Note:* We often use ω-notation to distinguish between terms in the sequel that are deterministic and those which are stochastic.
- At each time t, Agent j observes the correct value of S_t^{exc}, interprets this value as the realized value of $S_{j,\theta_j^\star,t}^{exc}$, and computes $S_{j,t}^{unf}$.

DG System

- That is, we get the identity

$$
\begin{aligned}
S_t^{exc}(\omega) &= S_{j,\theta_j^\star,t}^{exc}(\omega) \\
&= S_{j,t}^{unf}(\omega) + \eta_{j,per}\left(X_{j,t}^{\theta_j^\star}(\omega) - x_j\right) + \frac{1}{2}\eta_{j,tem}\theta_{j,t}^\star(\omega).
\end{aligned}
$$

- *Note:* From the *DG perspective*, $\theta_{j,t}^\star(\cdot)$ is evaluated at ω (it is evaluated at some $\omega_j \in \Omega_j$ from Agent j's perspective).

- Substituting S_t^{exc}'s dynamics into this identity implies that...

DG System

$$
\begin{aligned}
S_{j,t}^{unf}(\omega) - S_{j,0} = (S_0 - S_{j,0}) + \alpha t + \sum_{k \neq j} \tilde{\eta}_{k,per} \left(X_{k,t}^{\theta_k^\star}(\omega) - x_k \right) \\
+ \frac{1}{2} \sum_{k \neq j} \tilde{\eta}_{k,tem} \theta_{k,t}^\star(\omega) + \hat{W}_t(\omega) \\
+ (\tilde{\eta}_{j,per} - \eta_{j,per}) \left(X_{j,t}^{\theta_j^\star}(\omega) - x_j \right) \\
+ \frac{1}{2} (\tilde{\eta}_{j,tem} - \eta_{j,tem}) \theta_{j,t}^\star(\omega) .
\end{aligned}
$$

- *Note:* The LHS appears in Agent j's trading strategy!

DG System

- *Recall:* Agent j's trading strategy satisfies

$$\theta^{\star}_{j,t}(\omega) = -\sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth(\tau_j(t))\, X^{\theta^{\star}_j}_{j,t}(\omega) + \frac{\tanh(\tau_j(t)/2)\left[\mu_j + \nu_j^2\left(S^{unf}_{j,t}(\omega) - S_{j,0}\right)\right]}{\sqrt{\eta_{j,tem}\kappa_j}\left(1+\nu_j^2 t\right)}. \quad (8)$$

- *Notation:* Define the function $\Phi_j(\cdot)$ on $[0, T]$ by

$$\Phi_j(t) \triangleq \frac{\tanh(\tau_j(t)/2)\,\nu_j^2}{\sqrt{\eta_{j,tem}\kappa_j}\left(1+\nu_j^2 t\right)}. \quad (11)$$

DG System

- By substituting our expression for $S_{j,t}^{unf}(\omega) - S_{j,0}$ into the formula for Agent j's in (8), we get

$$\theta_{j,t}^{\star}(\omega)\left[1-\frac{1}{2}\left(\tilde{\eta}_{j,tem}-\eta_{j,tem}\right)\Phi_j(t)\right]-\frac{1}{2}\Phi_j(t)\sum_{k\neq j}\tilde{\eta}_{k,tem}\theta_{k,t}^{\star}(\omega)$$

$$=X_{j,t}^{\theta_j^{\star}}(\omega)\left[\left(\tilde{\eta}_{j,per}-\eta_{j,per}\right)\Phi_j(t)-\sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth\left(\tau_j(t)\right)\right]$$

$$+\Phi_j(t)\sum_{k\neq j}\tilde{\eta}_{k,per}X_{k,t}^{\theta_k^{\star}}(\omega)+\Phi_j(t)\left[\frac{\mu_j}{\nu_j^2}+\left(S_0-S_{j,0}\right)\right.$$

$$\left.+\alpha t+\hat{W}_t(\omega)-\sum_{k\neq j}\tilde{\eta}_{k,per}x_k-x_j\left(\tilde{\eta}_{j,per}-\eta_{j,per}\right)\right].$$

DG System

- *Note:* $\Phi_j \equiv 0$ when $\nu_j^2 = 0$, so in this case,

$$\theta^{\star}_{j,t} = -X^{\theta^{\star}_j}_{j,t}\sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth\left(\tau_j\left(t\right)\right) + \frac{\mu_j \tanh\left(\tau_j\left(t\right)/2\right)}{\sqrt{\eta_{j,tem}\kappa_j}}.$$

- *Note:* Agent j's strategy would be *deterministic* and *independent* of other agents' trading.

- *Intuition:* This would still be true, if we posited alternative dynamics for S^{exc} as Agent j does *not* learn by observing the realized price.

DG System

- By substituting Agent j's trading rate into the previous expression and rearranging, we see that the uncertain agents' strategies are characterized by the ODE system

$$\begin{aligned} A(t)\,\theta_t^{u,\star}(\omega) &= B(t)\,X_t^{u,\theta^\star}(\omega) + C(t,\omega) \\ X_0^{u,\theta^\star}(\omega) &= x^u. \end{aligned} \tag{12}$$

- $A(t) \in M_K(\mathbb{R})$ is given by

$$A_{ik}(t) \triangleq \begin{cases} 1 - \frac{1}{2}\left(\tilde{\eta}_{i,tem} - \eta_{i,tem}\right)\Phi_i(t) & \text{if } i = k \\ -\frac{1}{2}\tilde{\eta}_{k,tem}\Phi_i(t) & \text{if } i \neq k \end{cases}.$$

Coefficients

- $B(t) \in M_K(\mathbb{R})$ is given by

$$B_{ik}(t) \triangleq \begin{cases} (\tilde{\eta}_{i,per} - \eta_{i,per}) \Phi_i(t) - \sqrt{\dfrac{\kappa_i}{\eta_{i,tem}}} \coth(\tau_i(t)) & \text{if } i = k \\ \tilde{\eta}_{k,per} \Phi_i(t) & \text{if } i \neq k \end{cases} .$$

- $C(t,\omega) \in \mathbb{R}^K$ is given by

$$C_i(t,\omega) \triangleq \Phi_i(t) \left[c_i(t) + \tilde{W}_t(\omega) \right]$$

for some continuously differentiable map c_i.

Coefficients

- The key observations about these maps are as follows:
  - Φ_j is a strictly decreasing nonnegative function on $[0, T]$ with $\Phi_j(T) = 0$.
  - The entries of A are analytic on $[0, T]$ and $A(T) = I_K$.
  - The entries of B are analytic on $[0, T)$ but
$$\lim_{t \uparrow T} B_{jj}(t) = -\infty.$$
  - $C(\cdot, \omega)$'s entries are continuous on $[0, T]$ and $C(T, \omega) = 0$.
- Hence, if $\det A$ does not have a root on $[0, T]$, $S^{exc}(\omega)$, the $X_j^{\theta_j^\star}(\omega)$'s and the $\theta_j^\star(\omega)$'s are all uniquely defined and continuous on $[0, T)$.
- A little more work yields that in fact $X_j^{\theta_j^\star}(T) = 0$ (when $\det A$ does not have a root).

Key Problems

$$A(t)\,\vec{\theta}^{\star}_t(\omega) = B(t)\,\vec{X}^{\theta^\star}_t(\omega) + C(t,\omega)$$
$$\vec{X}^{\theta^\star}_0(\omega) = \vec{x}. \tag{12}$$

- *Problem 1:* $A(\cdot)$ may *not* be invertible on $[0, T]$, potentially causing solution existence and/or uniqueness to fail for (12).
- *Problem 2:* The diagonal entries of $B(\cdot)$ are *singular* at T, as they are of the form

$$(\tilde{\eta}_{j,per} - \eta_{j,per})\,\Phi_j(t) - \sqrt{\frac{\kappa_j}{\eta_{j,tem}}}\coth(\tau_j(t))\,.$$

Singularity

- *Notation:* If $\det A(\cdot)$ has a root on $[0, T]$, we can pick the smallest one and denote it by t_e.
- If $t_e > 0$ and $\omega \in \hat{\Omega}$ is such that $\hat{W}_\cdot(\omega)$ is continuous on $[0, t_e)$, then $S_\cdot^{exc}(\omega)$, the $X_{j,\cdot}^{\theta_j^\star}(\omega)$'s, and the $\theta_{j,\cdot}^\star(\omega)$'s are all uniquely defined and continuous on $[0, t_e)$.
- This follows by applying a standard existence/uniqueness theorem to

$$\vec{\theta}_t^\star(\omega) = A^{-1}(t)\left[B(t)\vec{X}_t^{\theta^\star}(\omega) + C(t,\omega)\right]$$

$$\vec{X}_0^{\theta^\star}(\omega) = \vec{x}. \tag{13}$$

- *Difficulty:* If $\det A(\cdot)$ has a root and $t_e > 0$, what happens as $t \uparrow t_e$?
- Near t_e, we consider the homogeneous equation

$$\vec{\theta}_t^\star(\omega) = A^{-1}(t)\, B(t)\, \vec{X}_t^{\theta^\star}(\omega) . \tag{14}$$

- We show that under certain conditions, we can find an analytic $M_K(\mathbb{R})$-valued function $D(\cdot)$, such that we can rewrite (14) near t_e as

$$\vec{\theta}_t^\star(\omega) = \frac{D(t)}{t - t_e} \vec{X}_t^{\theta^\star}(\omega) . \tag{15}$$

Singularity

- *Note:* When the representation in (15) exists, our homogeneous equation is said to have a *regular* singular point or a singular point of the *first kind* at t_e.

- *Note:* We can find some analytic $M_K(\mathbb{R})$-valued function $P(\cdot)$ with $P(t_e) = I_K$ such that for some small $\varepsilon > 0$, the fundamental solution of (15) on $(t_e - \varepsilon, t_e)$ is

$$P(t)\left|t - t_e\right|^{D(t_e)} \tag{16}$$

(Coddington et al., 1997).

- We can show that $D(t_e)$ is actually nicely behaved, e.g., it has at most one nonzero eigenvalue $\lambda \in \mathbb{R}$ and the geometric multiplicity of the eigenvalue 0 is (at least) $K - 1$.

Back to non-homogenous

$$\vec{\theta}_t^{\star}(\omega) = A^{-1}(t)\left[B(t)\,\vec{X}_t^{\theta^\star}(\omega) + C(t,\omega)\right] \tag{13}$$

- *Note:* Using the fundamental solution near t_e, we can solve (13) by *variation of parameters*.

General form of the solution

We can then find real constants

$$\{d_1(\omega), \ldots, d_K(\omega)\}$$

and real-valued continuous functions

$$\{F_1(\cdot,\omega), \ldots, F_K(\cdot,\omega)\}$$

such that near t_e, $\vec{X}_t^{\theta^\star}(\omega)$ is given by

$$P(t)\left[\sum_{j=1}^{K-1}\left(d_j(\omega) - \int_{t_e-\varepsilon}^{t} \frac{F_j(s,\omega)}{|s-t_e|}\,ds\right)v_j \right.$$
$$\left. + |t-t_e|^{\lambda}\left(d_K(\omega) - \int_{t_e-\varepsilon}^{t} \frac{F_K(s,\omega)}{|s-t_e|^{1+\lambda}}\,ds\right)v_K\right]. \quad (17)$$

Semi-Symmetric Uncertain Agent s

- *Definition:* The uncertain agents are *semi-symmetric* when there are positive constants

$$\tilde{\eta}_{tem}, \quad \eta_{tem}, \quad \tilde{\eta}_{per}, \quad \eta_{per}, \quad \nu^2, \quad \text{and} \quad \kappa$$

such that for each $i \in \{1, \ldots, K\}$

$$\begin{array}{lll} \tilde{\eta}_{i,tem} = \tilde{\eta}_{tem}, & \eta_{i,tem} = \eta_{tem}, & \tilde{\eta}_{per} = \tilde{\eta}_{i,per}, \\ \eta_{i,per} = \eta_{per}, & \nu_i^2 = \nu^2, & \kappa_i = \kappa. \end{array}$$

- *Note:* Semi-symmetric agents differ in their initial inventories x_j and the mean μ_j of their initial drift estimates.

- Assume that the uncertain agents are semi-symmetric (no constraints on the rest).

Semi-Symmetric Uncertain Agent s

- One can show that $\det A(\cdot)$ has a root on $(0, T)$ if and only if

$$\left(L\tilde{\eta}_{tem} - \eta_{tem}\right)\Phi(0) > 2. \tag{18}$$

- In this case, our homogeneous equation *always* has a regular singular point at t_e.

- Moreover,

$$\begin{aligned}
\{\lambda > 0\} \quad &\iff \quad \left\{\frac{2\left(L\tilde{\eta}_{per} - \eta_{per}\right)}{L\tilde{\eta}_{tem} - \eta_{tem}} - \sqrt{\frac{\kappa}{\eta_{tem}}}\coth\left(\tau\left(t_e\right)\right) > 0\right\} \\
\{\lambda < 0\} \quad &\iff \quad \left\{\frac{2\left(L\tilde{\eta}_{per} - \eta_{per}\right)}{L\tilde{\eta}_{tem} - \eta_{tem}} - \sqrt{\frac{\kappa}{\eta_{tem}}}\coth\left(\tau\left(t_e\right)\right) < 0\right\}.
\end{aligned}$$

Semi-Symmetric Uncertain Agents

- If $\lambda < 0$, then

$$\left\{ d_L(\omega) - \lim_{t \uparrow t_e} \left[\int_{t_e - \rho}^{t} \frac{F_L(s,\omega)}{|s - t_e|^{1+\lambda}} ds \right] > 0 \right\}$$

$$\Longrightarrow \quad \left\{ \lim_{t \uparrow t_e} X_{j,t}^{\theta_j^\star}(\omega) = \lim_{t \uparrow t_e} \theta_{j,t}^\star(\omega) = +\infty, \quad \text{for } j = 1, \ldots, L \right\}$$

and

$$\left\{ d_L(\omega) - \lim_{t \uparrow t_e} \left[\int_{t_e - \rho}^{t} \frac{F_L(s,\omega)}{|s - t_e|^{1+\lambda}} ds \right] < 0 \right\}$$

$$\Longrightarrow \quad \left\{ \lim_{t \uparrow t_e} X_{j,t}^{\theta_j^\star}(\omega) = \lim_{t \uparrow t_e} \theta_{j,t}^\star(\omega) = -\infty, \quad \text{for } j = 1, \ldots, L \right\}.$$

$S_t^{exc}(\omega)$ will explode in the direction of the agents' trades!

Semi-Symmetric Uncertain Agent s

- *Note:* When $\lambda > 0$ then inventories are finite, but rates explode and stock price explodes in the direction of the trades.
- *Note:* The time of a potential mini flash crash is *deterministic.* however, the direction is *random*.
- *Note:* Whether the price will explode up or down is *random*.
- *Note:* A mini flash crash can only occur if *at least* one agent is uncertain.

Semi-Symmetric Uncertain Agent s

- *Intuition:* Since a mini flash crash can only occur if

$$
\begin{aligned}
2 &< \left(K\tilde{\eta}_{tem} - \eta_{tem}\right)\Phi\left(0\right) \\
&= \left(K\tilde{\eta}_{tem} - \eta_{tem}\right)\left[\frac{\tanh\left(\sqrt{\frac{\kappa}{\eta_{tem}}}T/2\right)\nu_j^2}{\sqrt{\eta_{tem}\kappa}}\right],
\end{aligned}
$$

factors that contribute to such an occurrence

- low agent confidence in model drift parameters;
- agent underestimation of temporary market impact;
- large agent populations;
- long trading horizons.
- low inventory penalty.

Thank you for your attention!