Proactive and reactive trading: Optimal control with Meyer σ-fields

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Information flow and optimal control

In many financial optimal control problems, one can think of moments where significant new information is known to become available:

- interest rate decisions by central banks, elections, referendums
- publication of data on GDP growth, job market statistics, trade balances
- price jumps, e.g., at opening of exchanges, from earning announcements, ...
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How to describe such information flows mathematically?
How to do optimal control with them?

Illustrative control problem: Irreversible investment

- Classic problem: Dixit and Pindyck (1994), Bertola (1998), Riedel and Su (2011), Ferrari (2015), ...
- A firm manager has to decide about the level of installed capacity C_t at any time $t \geq 0$. With installed capacity c, she gets a unit price $\gamma_t(c)$ for the current demand dD_t; expanding capacity at time t by dC_t costs $\xi_t\, dC_t$; investment is irreversible, i.e., C is increasing.
  Hence, her expected net proceeds are

$$\mathbb{E}\left[\int_{[0,\infty)} \gamma_t(C_t)\, dD_t - \int_{[0,\infty)} \xi_t\, dC_t\right] \to \max_{C \geq c_0 \nearrow \text{ l.c., adapted}}.$$

- *Typical* assumptions: $\xi > 0$ super-martingale deflator;
  γ exhibits decreasing returns to scale, e.g., $\gamma_t(c) = e^{-rt} \log c$;
  $dD_t \ll dt$, e.g., $dD_t = dt$

Solution via representation theorem

If dD is atomless with full support and $\gamma_t(\ell) \in L^1(\mathbb{P} \otimes dt)$ then a deflator ξ of class (D) with $\xi_\infty = 0$ can be written in the form

$$\xi_t = \mathbb{E}\left[\int_{(t,\infty)} \partial_c \gamma_u\left(\sup_{v \in [t,u)} L_v\right) dD_u \,\middle|\, \mathscr{F}_t\right], \quad t \geq 0,$$

for a suitable optional $L \geq 0$; cf. B.-El Karoui (2004).

Hence, the optimal left-cont. control is $\hat{C}_t \triangleq \sup_{v \in [0,t)} L_v$

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$$\begin{aligned}
&\mathbb{E}\Big[\int_{[0,\infty)} \gamma_u(C_u)\, dD_u - \int_{[0,\infty)} \xi_t\, dC_t\Big] \\
&= \mathbb{E}\Big[\int_{[0,\infty)} \Big\{\gamma_u(C_u) - \int_{[0,u)} \partial_c \gamma_u\big(\underbrace{\sup_{v\in[t,u)} L_v}_{\leq \hat{C}_u}\big)\, dC_t\Big\} dD_u\Big] \\
&\leq \mathbb{E}\Big[\int_{[0,\infty)} \underbrace{\{\gamma_u(C_u) - \partial_c \gamma_u(\hat{C}_u) C_u\}}_{\leq \text{ same with } C_u \text{ replaced by } \hat{C}_u}\, dD_u\Big]
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\end{aligned}$$

What if dD allows for point masses: **“demand surges”**?

Capacity expansions with jumps: action vs. reaction

- A demand surge $\Delta D_t > 0$ gives an incentive to have an increased capacity at time t to collect

$$\gamma_t(C_t)\Delta D_t.$$

  Taking into account possible price changes are expected, expanding capacity at this point may make sense: $C_t > C_{t-}$

- The actual demand/price change may only be fully known afterwards and may incentivize a further capacity expansion: $C_{t+} > C_t$

$\rightsquigarrow$ Possibly two capacity expansions – at possibly different prices:

$$\xi_t \, dC_t^c + \xi_t(C_t - C_{t-}) + \xi_{t*}(C_{t+} - C_t) =: \xi_{t\,*} dC_t$$

with $\xi_{t*} := \liminf_{u \downarrow t} \xi_u$; cf. Lenglart (1980), Campi-Schachermayer (2006), Guasoni et al. (2012)

Same toy model - Different information flows

Example: *demand*: $dD_t = 1_{[0,2]}(t)dt + \delta_1(dt) \rightsquigarrow$ surge in $t = 1$;
capacity prices (ξ_t) are at level 60 on $[0, 1]$ and then

- continue to be 60 on $(1, 2]$ in worst price scenario ω_1;
- jump to and stay at 40 on $(1, 2]$ in OK price scenario ω_2;
- jump to and stay at 30 on $(1, 2]$ in best price scenario ω_3;

Right-cont. observation filtration $(\mathscr{F}_t)$ is naturally generated by ξ.
But capacity controls C can be restricted to be ...

... predictable: capacity expansion at time 1 with **no knowledge** of next prices

... optional: capacity expansion at time 1 with **full knowledge** of next prices

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... "in between": capacity expansion at time 1 with **some knowledge** of next prices:

- *either* "Will prices remain high?" - Yes/No
- *or* "Will the best price become available?" - Yes/No
- *or* "Will the OK level price obtain afterwards?" - Yes/No

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Unifying framework: **Meyer σ-fields** Λ with $\mathscr{P} \subset \Lambda \subset \mathscr{O}$

Meyer σ-fields Λ: Definition and some properties

A σ-field on $\Omega \times [0, \infty)$ is called a **Meyer σ-field** if

- it is generated by càdlàg processes;
- it contains all deterministic Borel-measurable events;
- it is stable with respect to stopping: with Z also $(Z_{s\wedge t})_{s\geq 0}$ is Λ-measurable for any $t \geq 0$.

A Meyer-σ-field induces ...

... a set of **stopping times**: $\mathscr{S}^\Lambda \ni S$ iff $[S, \infty[\in \Lambda$;

... a **filtration** via $\mathscr{F}_S^\Lambda := \sigma(Z_S \mid Z\ \Lambda\text{-measurable}),\ S \in \mathscr{S}^\Lambda$;

Meyer's section theorem applies: For any $B \in \Lambda$ and any $\varepsilon > 0$, there is $S \in \mathscr{S}^\Lambda$ with

$$(\omega, S(\omega)) \in B \text{ whenever } S(\omega) < \infty$$

and

$$\mathbb{P}[S < \infty] \geq \mathbb{P}[\mathrm{proj}_\Omega(B)] - \varepsilon.$$

Moreover, there are Λ-**projections**, super-martingales, tests for path properties...

General irreversible investment problem

Maximize

$$\mathbb{E}\left[\int_{[0,\infty)} \gamma_t(C_t)\,dD_t - \int_{[0,\infty)} \xi_{t\,*}dC_t\right]$$

over Λ-measurable increasing (only làdlàg) controls C where

- dD optional measure possibly with random atoms, not necessarily with full support; e.g., $dD_t = 1_{[0,2]}(t)dt + \delta_1(dt)$
- ξ is Λ-measurable with $\xi_S = 0$ if $dD([S,\infty)) = 0$ a.s., $S \in \mathscr{S}^\Lambda$; e.g., ξ from above toy example
- $(t,\omega) \mapsto \gamma_t(c)(\omega)$ progressively measurable, $c \mapsto \gamma_t(c)$ concave, suitable Inada conditions; e.g., $\gamma_t(c) = \log c$

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Solution via stochastic representation theorem. . .

Assumptions and some terminology

The process $D = (D_t)$ is right-continuous and adapted;
$g = g_t(\omega, \ell) : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ (think $-\partial_c \gamma_t(\ell)$) is progressively measurable in (ω, t) for any fixed $\ell \in \mathbb{R}$ with $\mathbb{E} \int_{[0,\infty)} |g_t(\ell)| dD_t < \infty$ and continuous and strictly increasing from $-\infty$ to $+\infty$ in ℓ for any fixed $(\omega, t) \in \Omega \times [0, \infty)$.

A Λ-measurable process X (think $-\xi$) is called ...

... **of class($\mathbf{D}^\Lambda$)** if $\{X_S \mid S \in \mathscr{S}^\Lambda\}$ is uniformly integrable;

... dD-**upper-semicontinuous** if

$$\limsup_n \mathbb{E}[X_{S_n}] \leq \mathbb{E}[X_S]$$

for any sequence of stopping times (S_n) such that

- *either* each S_n is predictable with $S_n \uparrow S$ on $\{S > 0\}$,
- *or* each S_n is in $\mathscr{S}^\Lambda([S, \infty])$ with $dD([S, S_n)) \to 0$;

Representation Theorem

Theorem

Any dD-upper-semicontinuous Λ-measurable process X of class(D^Λ) such that $X_S = 0$ for $S \in \mathscr{S}^\Lambda$ with $dD([S,\infty)) = 0$ a.s. is of the form

$$X_S = \mathbb{E}\left[\int_{[S,\infty)} g_t(\sup_{v\in[S,t]} L_v)dD_t \,\middle|\, \mathscr{F}_S^\Lambda\right], \quad S \in \mathscr{S}^\Lambda,$$

for some Λ-measurable process L such that $g(\sup_{v\in[S,.]} L_v)1_{[S,\infty)}$ is $\mathbb{P}\otimes dD$-integrable for any $S \in \mathscr{S}^\Lambda$. The maximal such process is

$$L_S = \operatorname*{ess\,inf}_{T\in\mathscr{S}^\Lambda((S,\infty])} \ell_{S,T}, \quad S \in \mathscr{S}^\Lambda,$$

where $\ell_{S,T} \in \mathscr{F}_S^\Lambda$ solves

$$\mathbb{E}\left[X_T - X_S \,\middle|\, \mathscr{F}_S^\Lambda\right] = \mathbb{E}\left[\int_{[S,T)} g_t(\ell)dD_t \,\middle|\, \mathscr{F}_S^\Lambda\right].$$

Illustration in toy example

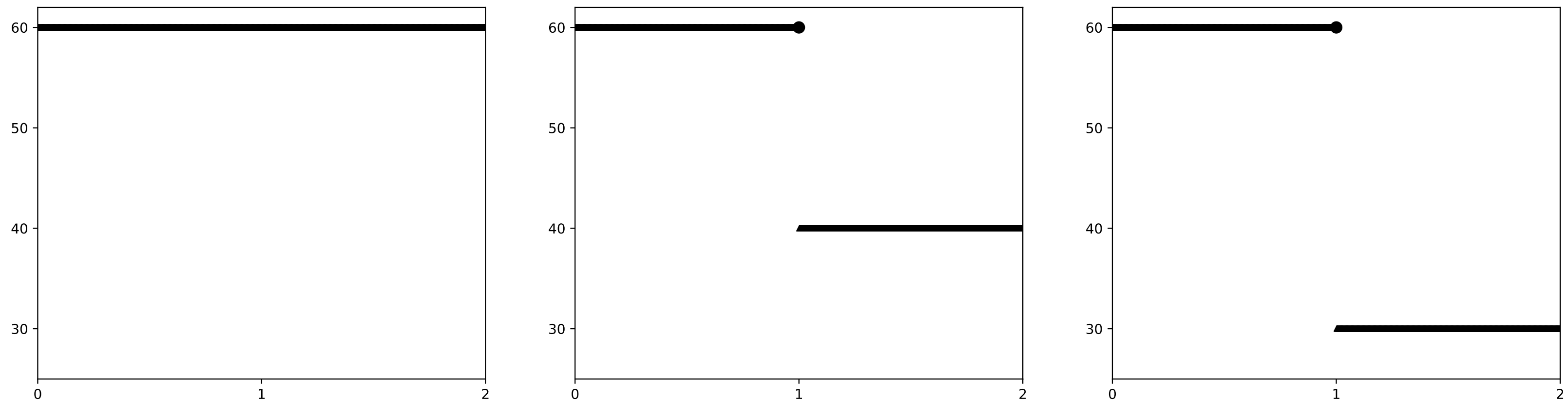
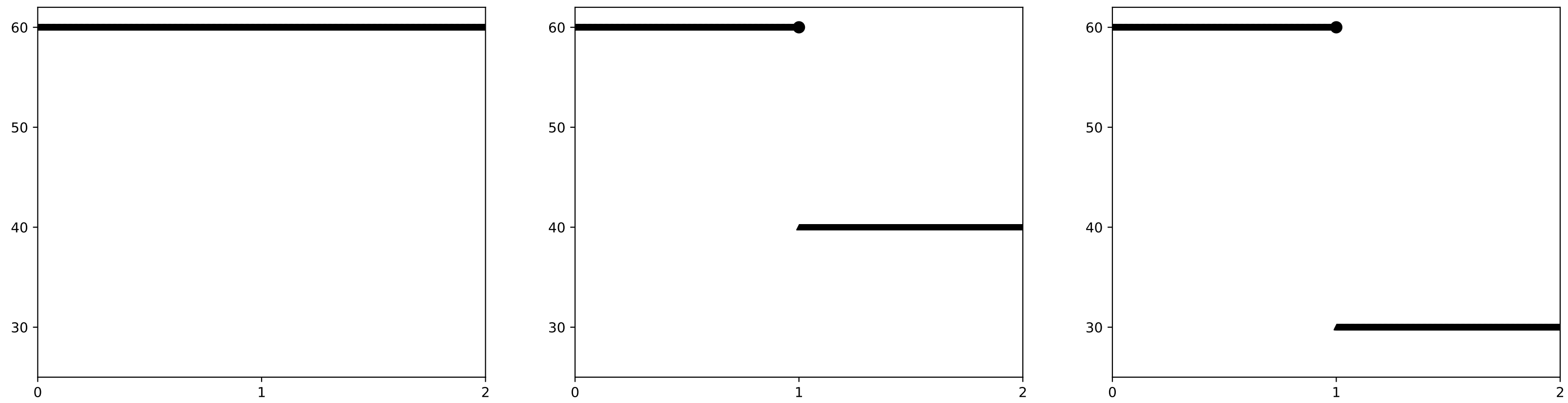


Figure: Price evolution of ξ in worst, OK, and best scenario, respectively.

Illustration in toy example



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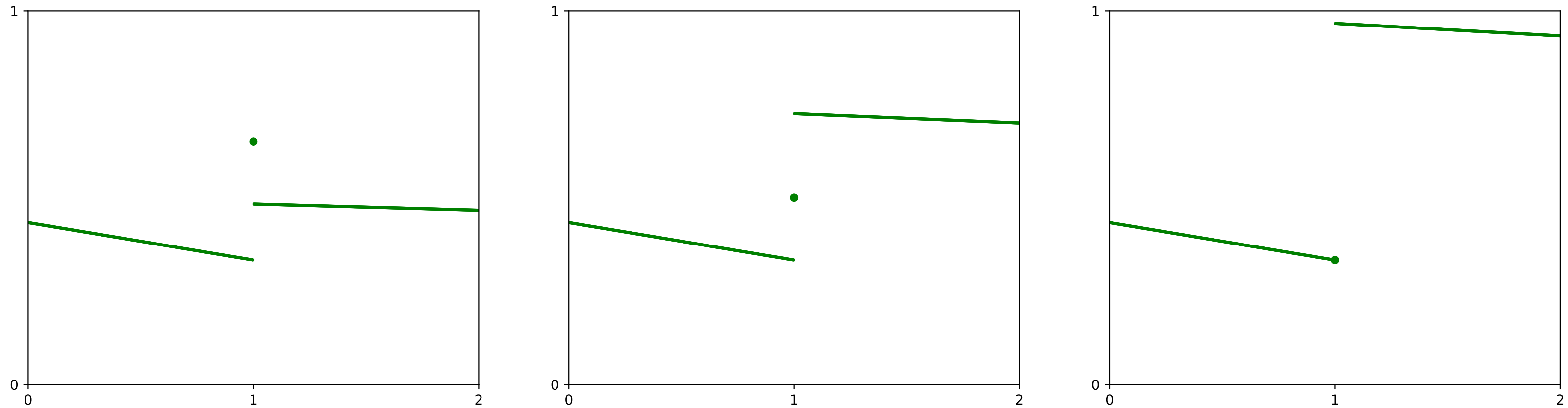
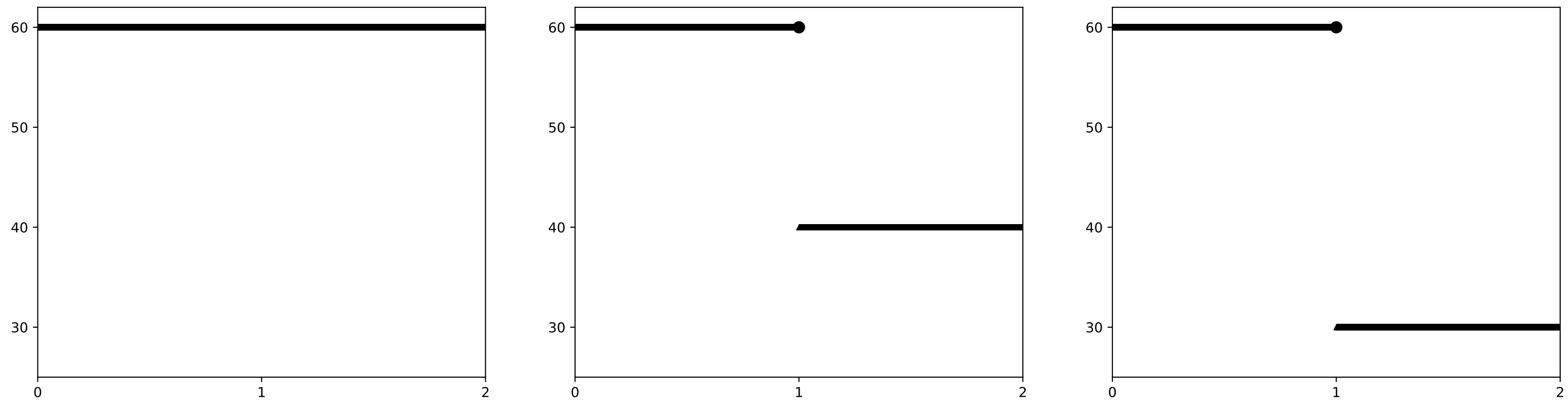


Figure: Solution $L^{\mathscr{O}}$ for **full knowledge** of next prices: optional case

Illustration in toy example



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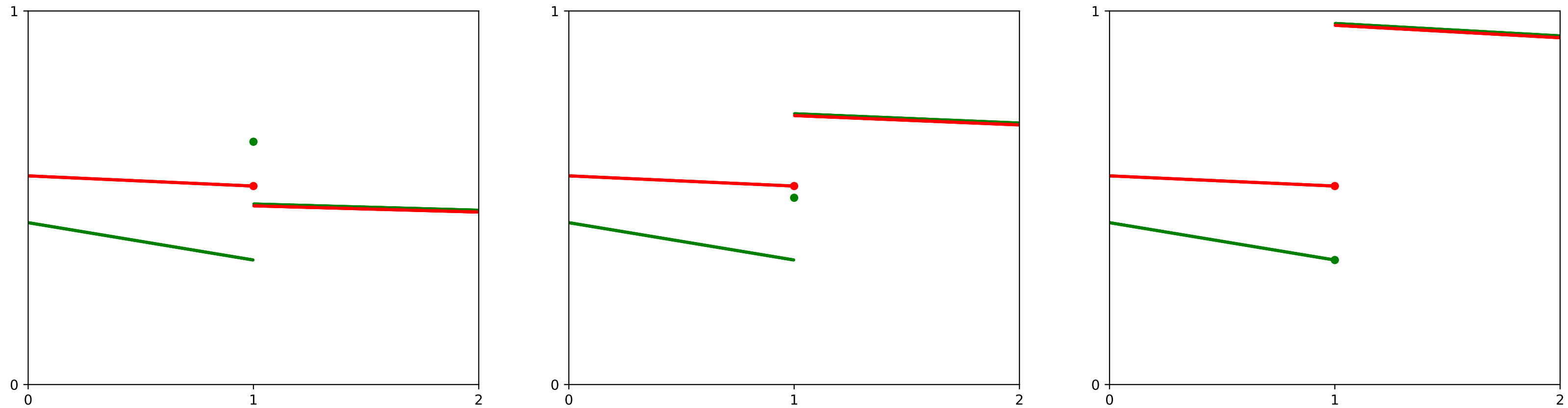
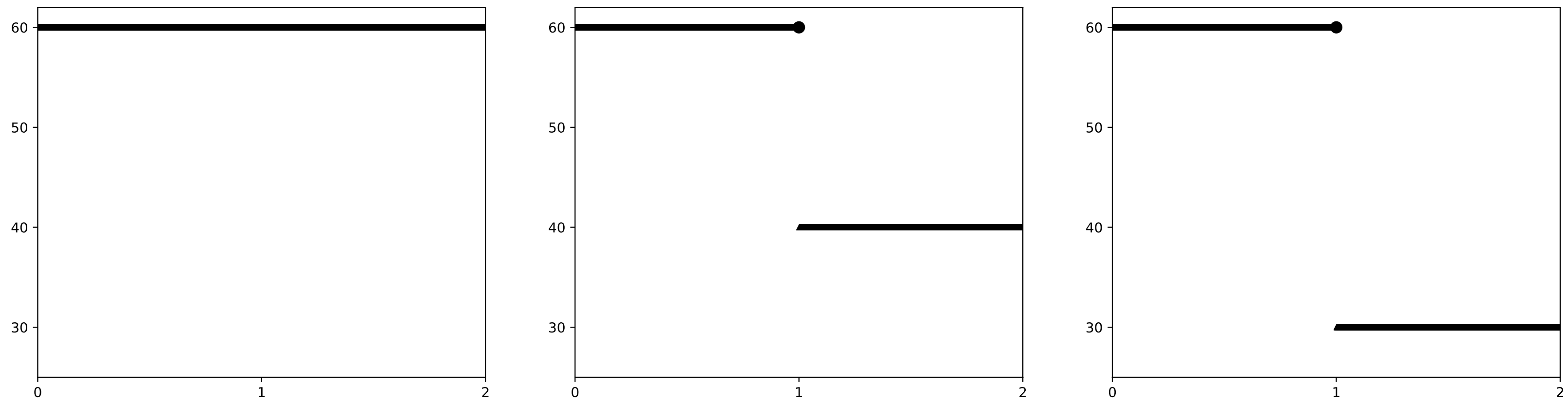


Figure: Solution $L^{\mathscr{P}}$ for **no knowledge** of next prices: predictable case

Illustration in toy example



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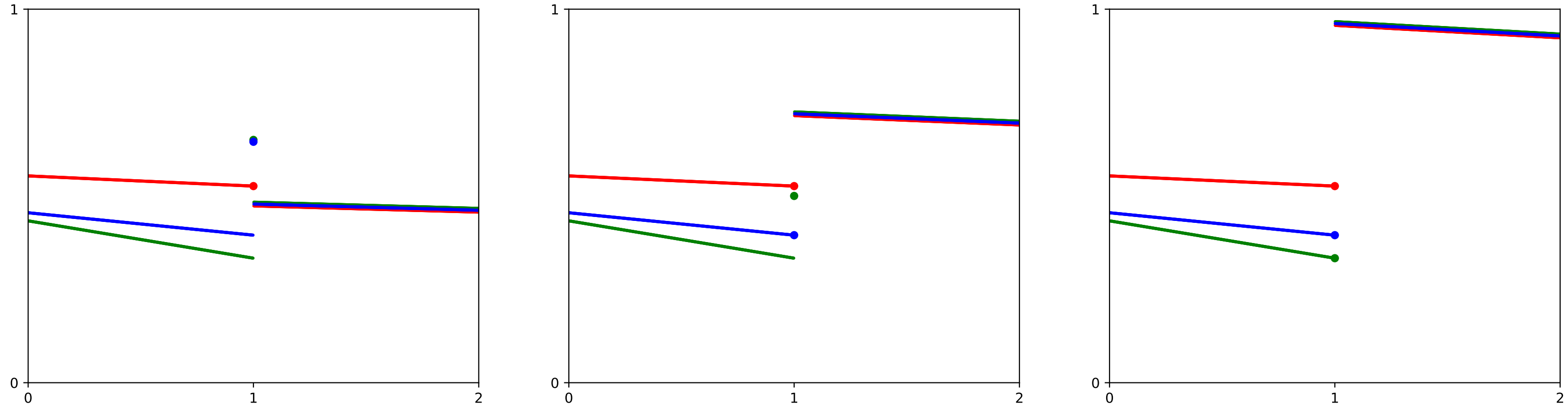


Figure: Solution L^{Λ} for **some knowledge** of next prices: Meyer case

Illustration in toy example

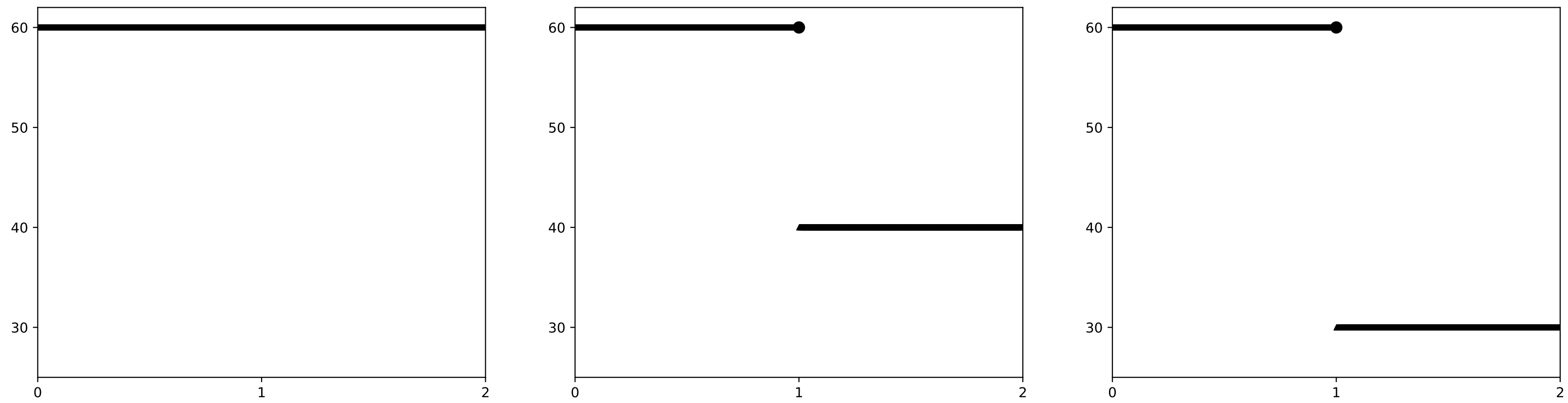


Figure: Price evolution of ξ in worst, OK, and best scenario, respectively.

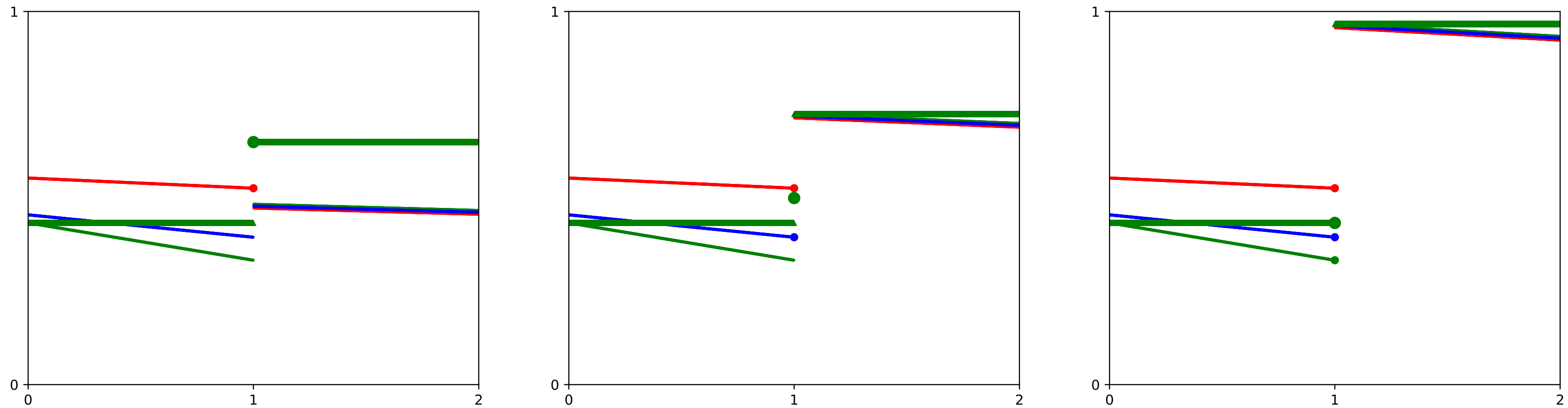


Figure: Optimal policy with **full knowledge** of next prices: optional case

Illustration in toy example

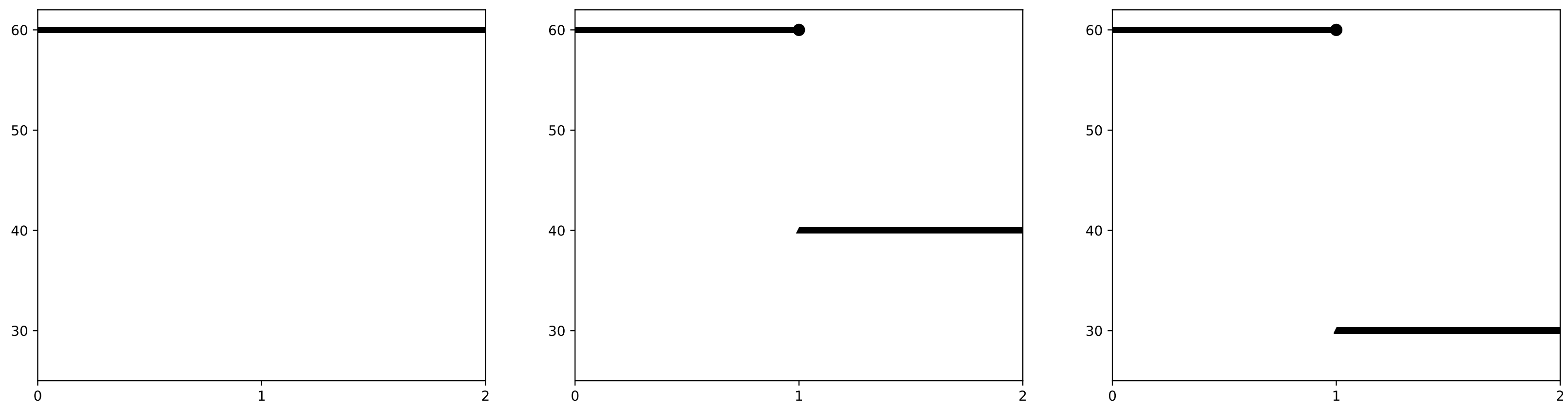


Figure: Price evolution of ξ in worst, OK, and best scenario, respectively.

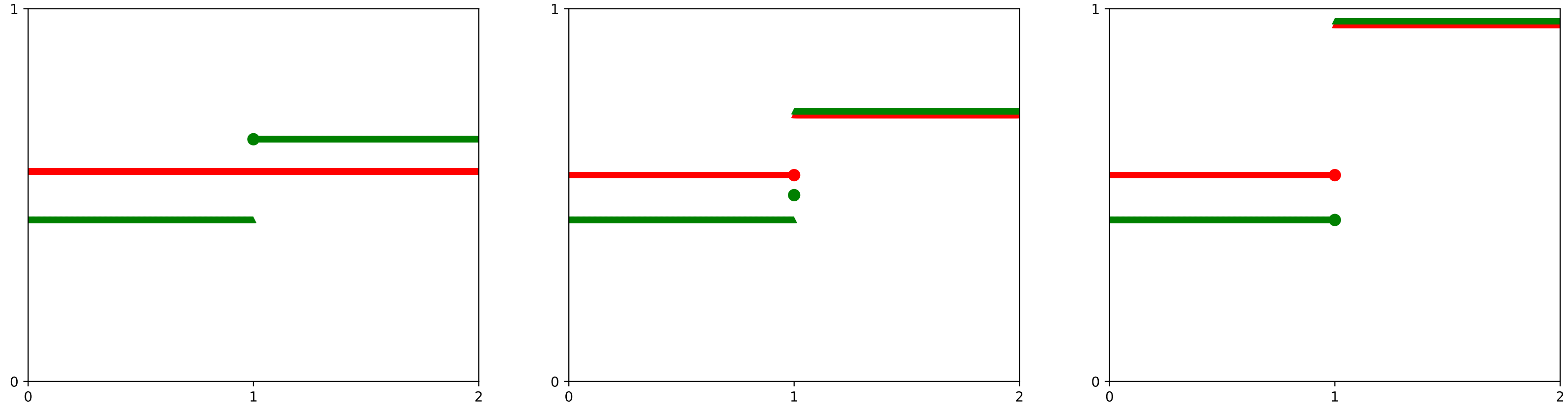
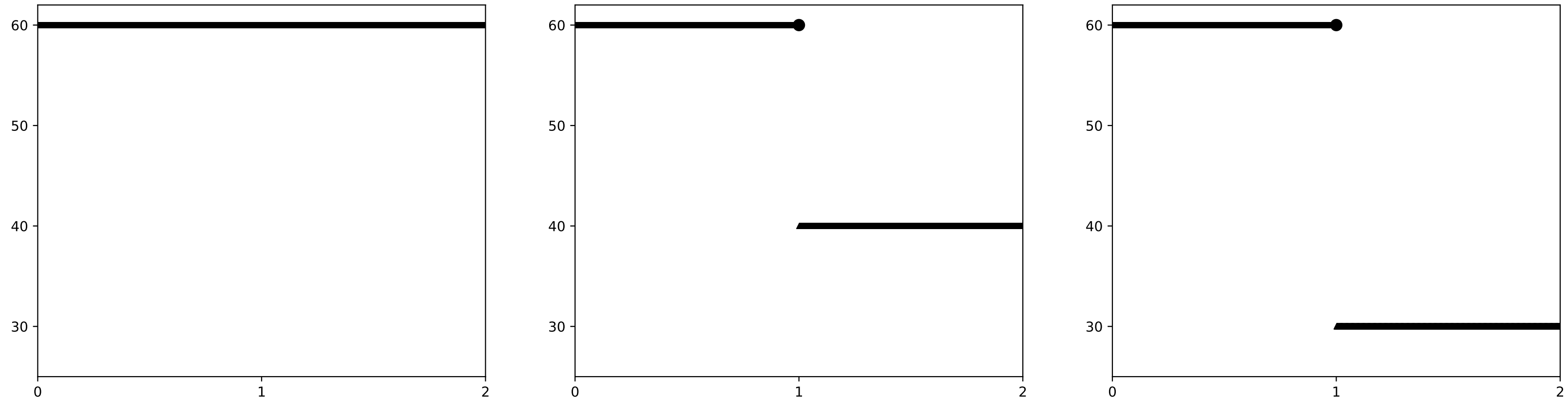


Figure: Optimal policy with **no knowledge** of next prices: predictable case

Illustration in toy example



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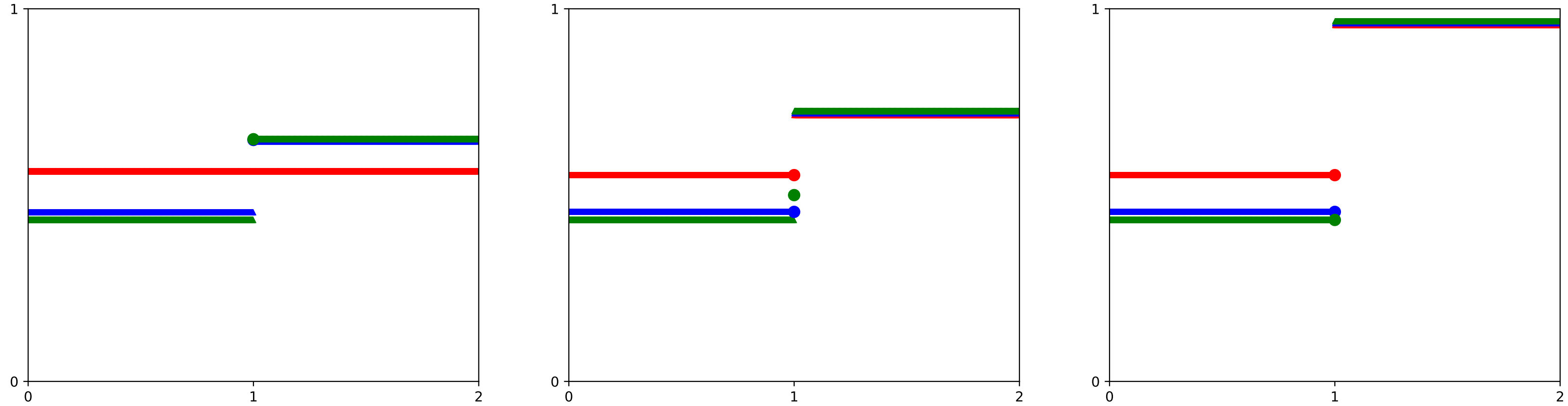


Figure: Optimal policy with **knowledge whether prices will improve**: Meyer case

Illustration: prices driven by compound Poisson process

$$\mathbb{E}\left[\int_{[0,\hat{T})} \xi_t{}^* dC_t - \int_{[0,\hat{T})} \gamma_t(C_t)\, dD_t\right] \to \max_{C \text{ increasing from } C_{0-}=c_0}$$

- $D_t \triangleq N_t,\ t \geq 0$, standard Poisson process with parameter λ_1
- ...counting the jumps of price process

$$\xi_t \triangleq \xi_0 + \sum_{n=1}^{N_t} Y_n, \quad t \geq 0,$$

  with i.i.d. $Y_1, Y_2, \ldots > 0$ ind. of N with $m \triangleq \mathbb{E} Y_1 < \infty$
- $\gamma_t(c) = \frac{1}{2}c^2,\ c \in \mathbb{R},\ \hat{T} \sim \mathrm{Exp}(\lambda_2)$ independent time horizon
- observation filtration $(\mathscr{F}_t)$ filtration generated by ξ and $\hat{T}$
- sensor warns about impending jumps if they are large enough:

$$\mathscr{P} \subset \Lambda^\eta \triangleq \mathscr{P} \vee \sigma(\chi^\eta) \subset \mathscr{O}$$

  where, for some detection threshold $\eta \geq 0$,

$$\chi_t^\eta \triangleq \sum_{n=1}^{N_t} Y_n 1_{\{Y_n > \eta\}}, \quad t \geq 0.$$

Illustration: prices driven by compound Poisson process

$$\mathbb{E}\left[\int_{[0,\hat{T})} \xi_t^{\eta\,*}\, dC_t - \int_{[0,\hat{T})} \gamma_t(C_t)\, dD_t\right] \to \max_{C \in \Lambda^\eta \text{ increasing from } C_{0-}=c_0}$$

- effective cost process for controller with information Λ^η:

$$\xi_t^\eta \triangleq {}^{\Lambda^\eta}\xi_t = \xi_{t-} + Y_{N_t} 1_{\{\Delta\xi_t > \eta\}}, \quad t \geq 0$$

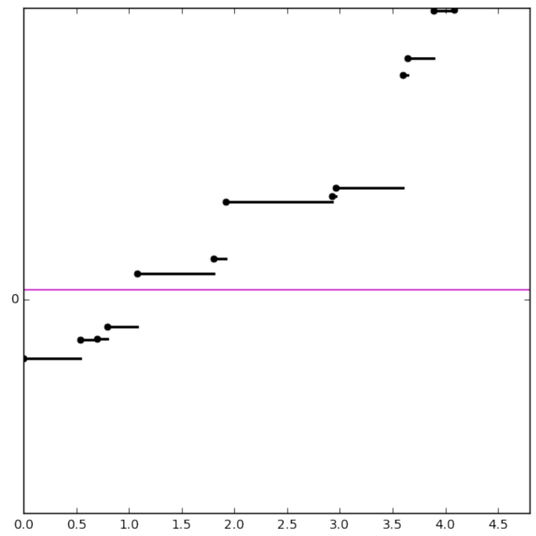
- solution to representation problem for ξ^η:

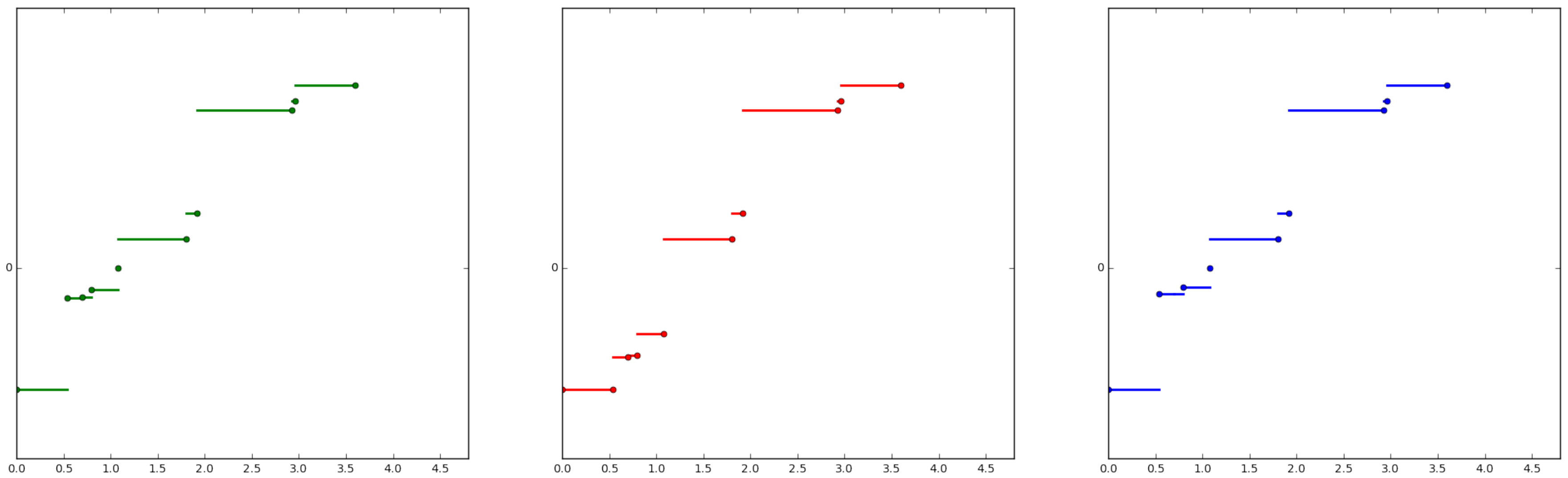
$$L_t^\eta = (\xi_t^\eta - m\frac{\lambda_1}{\lambda_2}) \cdot \begin{cases} 0, & \xi_t^\eta \geq \frac{\lambda_1}{\lambda_2} m, \Delta\xi_t > \eta \\ \frac{\lambda_2}{\lambda_1}, & \xi_t^\eta \geq \frac{\lambda_1}{\lambda_2} m, \Delta\xi_t \leq \eta \\ \frac{1}{1+\frac{\lambda_1}{\lambda_1+\lambda_2}p} \frac{\lambda_2}{\lambda_1+\lambda_2}, & \xi_t^\eta < \frac{\lambda_1}{\lambda_2} m, \Delta\xi_t > \eta \\ \frac{1}{p} \frac{\lambda_2}{\lambda_1+\lambda_2}, & \xi_t^\eta < \frac{\lambda_1}{\lambda_2} m, \Delta\xi_t \leq \eta \end{cases}$$

where $p \triangleq \mathbb{P}[Y_1 \leq \eta]$ is sensor's "failure-to-alert"-probability

Illustration: prices driven by compound Poisson process

Price evolution ξ
and threshold $m\lambda_1/\lambda_2$





optimal optional control optimal predictable control optimal Meyer control

Conclusions

- Information for control policies described by Meyer σ-fields
- Account in a very flexible manner for possible signals on information shocks and avoid delayed filtrations
- Illustration by general irreversible investment problem with signals on demand shocks
- Solution constructed from representation theorem for sufficiently regular Meyer processes
- Optimal controls are just ladlag in general: proactive and reactive control
- Explicit solutions in toy example and with some compound Poisson process
- Future work: explicit solutions with (more) general Lévy processes; signals with expected jumps; different target functionals like optimal order execution with stochastic liquidity; trading with transaction costs ...

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Thank you very much!