On the dynamics of Jeandel-Rao tilings

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CNRS + LaBRI + Université de Bordeaux

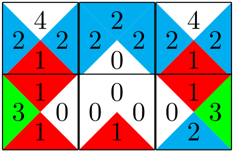
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z}^2 \to T$$

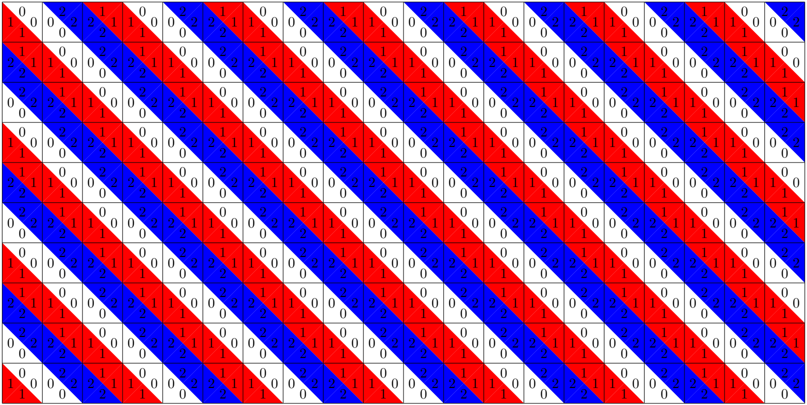
of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

A tile set is **finite** if there is no tiling of the plane with this set.
A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

Every set is either **finite or periodic**.

...shown later to be False :

Theorem (Berger 1966)

There **exists an aperiodic** set of Wang tiles.

Discoveries of aperiodic Wang tile sets (< 2015)

Image credit : `http://chippewa.canalblog.com/archives/2010/06/04/18115718.html`

- Berger : 20426 tiles in 1966 (lowered down later to 104)
- Knuth : 92 tiles in 1968
- Robinson : 56 tiles in 1971
- Ammann : 16 tiles in 1971
- Grunbaum : 24 tiles in 1987
- Kari : 14 tiles in 1996
- Culik : (same method) 13 tiles in 1996

ELSEVIER

Discrete Mathematics 160 (1996) 259–264

DISCRETE
MATHEMATICS

Note

A small aperiodic set of Wang tiles

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Received 3 January 1995

Abstract

A new aperiodic tile set containing only 14 Wang tiles is presented. The construction is based on Mealy machines that multiply Beatty sequences of real numbers by rational constants.

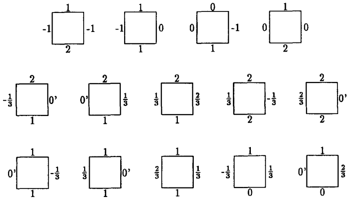


Fig. 1. Aperiodic set of 14 Wang tiles.

Proposition 1. *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f: \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period a and vertical period b. For $i \in \mathbb{Z}$, let n_i denote the sum of colors on the upper edges of tiles $f(1, i), f(2, i), \dots, f(a, i)$. Because the tiling is horizontally periodic with period a, the 'carries' on the left edge of $f(1, i)$ and the right edge of $f(a, i)$ are equal. Therefore $n_{i+1} = q_i n_i$, where $q_i = 2$ if tiles of T_2 are used on row i and $q_i = \frac{2}{3}$ if tiles of $T_{2/3}$ are used.

Because the vertical period of tiling f is b,

$$n_1 = n_{b+1} = q_1 q_2 \dots q_b \cdot n_1,$$

and because two tiles with 0's on their upper edges cannot be next to each other, $n_1 \neq 0$. So $q_1 q_2 \dots q_b = 1$. This contradicts the fact that no non-empty product of 2's and $\frac{2}{3}$'s can be 1. □

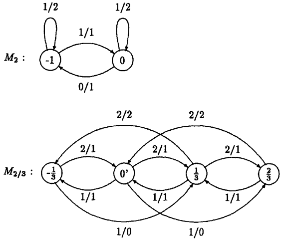


Fig. 2. Mealy machine corresponding to the aperiodic tile set.

The transducer approach (Jeandel, Rao, 2015)

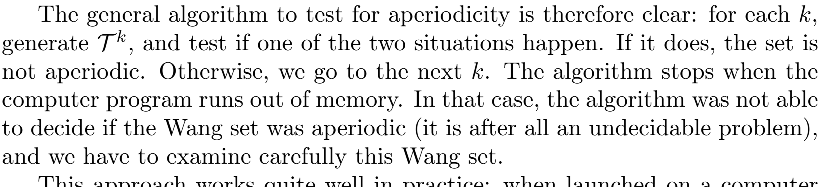
We identify a tile set $\mathcal{T}$ with its transducer (or its dual transducer).

Lemma

A Wang tile set $\mathcal{T}$ is **not aperiodic** if either

- (**finite**) there is k s.t. the str. conn. comp. of $\mathcal{T}^k$ is empty : i.e., there is no biinfinite words w, w' s.t. $w\mathcal{T}^k w'$.
- (**periodic**) or there exists k s.t. $\mathcal{T}^k$ is periodic : i.e., there is a biinfinite word w s.t. $w\mathcal{T}^k w$.

Jeandel, Rao (p. 8) :



The general algorithm to test for aperiodicity is therefore clear: for each k, generate $\mathcal{T}^k$, and test if one of the two situations happen. If it does, the set is not aperiodic. Otherwise, we go to the next k. The algorithm stops when the computer program runs out of memory. In that case, the algorithm was not able to decide if the Wang set was aperiodic (it is after all an undecidable problem), and we have to examine carefully this Wang set.

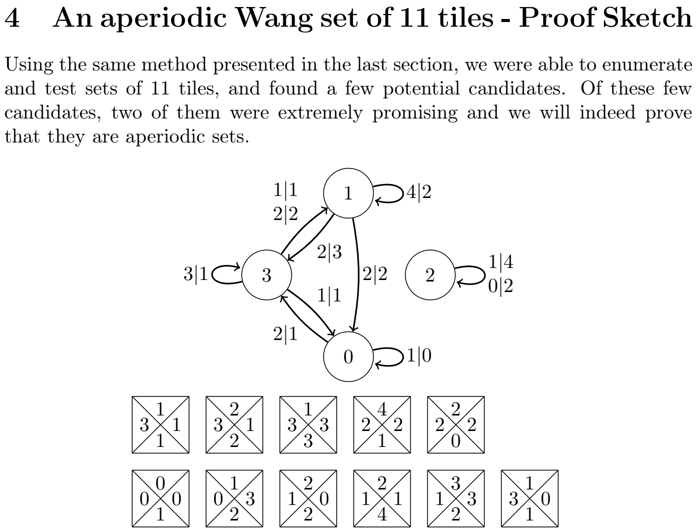
This approach works quite well in practice: when launched on a computer

Discoveries of aperiodic Wang tile sets ($\geq$ 2015)

- Jeandel, Rao : every set of $\leq$ 10 tiles is finite or periodic
- Jeandel, Rao : 11 tiles in 2015

Image credit : Le Bagger 288, `http://i.imgur.com/YH9xX.jpg`

Jeandel, Rao, `arxiv:1506.06492`

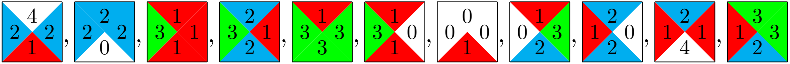


4 An aperiodic Wang set of 11 tiles - Proof Sketch

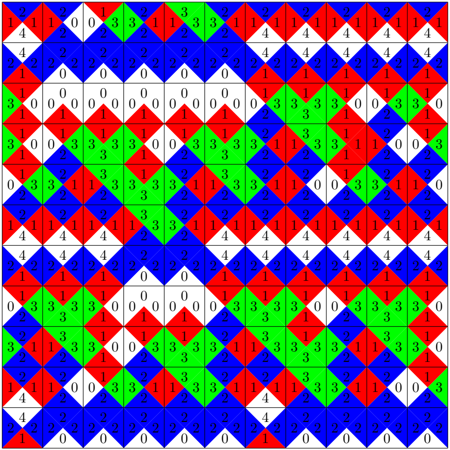
Using the same method presented in the last section, we were able to enumerate and test sets of 11 tiles, and found a few potential candidates. Of these few candidates, two of them were extremely promising and we will indeed prove that they are aperiodic sets.

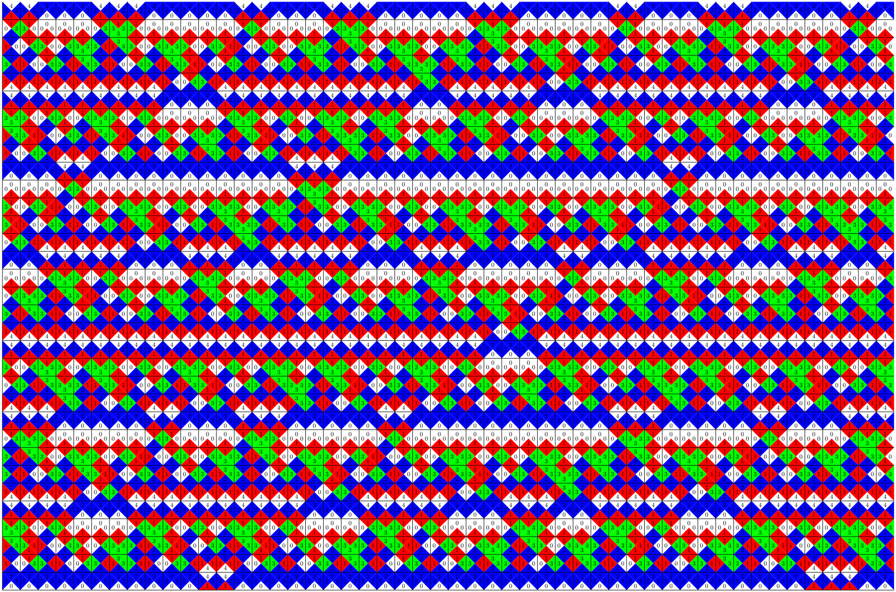
Jeandel-Rao 11 tiles set

$\mathcal{T} = \{$

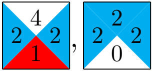


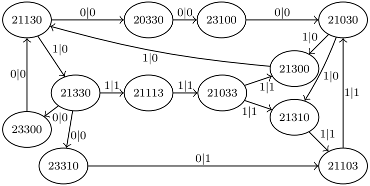
$\}$.

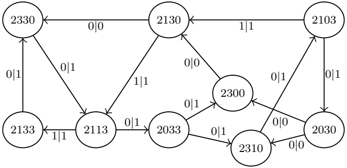


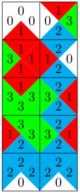


Some observations made by Jeandel, Rao

- horizontal lines are over $\mathcal{T}_1 = \Bigg\{$  $\Bigg\}$ or $\mathcal{T}_0 = \Bigg\{$  $\Bigg\}$
- The str. conn. comp. of the product $\mathcal{T}_1\mathcal{T}_1$, $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_1$, $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_1$ and $\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$ **are empty**.
- Every tiling by $\mathcal{T}$ can be **decomposed** into a tiling by transducers $\mathcal{T}_a = \mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$ and $\mathcal{T}_b = \mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$.
- Every tiling can be desubstituted uniquely by the **31 patterns** of rectangular shape 1×4 or 1×5 associated to the edges of $\mathcal{T}_a$ and $\mathcal{T}_b$:







Left image credit : `arxiv:1506.06492` , Figure 7 (b) at page 17.

An aperiodic set of 11 Wang tiles

Proposition (Jeandel, Rao, 2015)

The Wang set $\mathcal{T}_a \cup \mathcal{T}_b$ is **aperiodic**. Furthermore, the set of words $u \in \{a,b\}^*$ s.t. the sequence of transducers $\mathcal{T}_{u_1} \dots \mathcal{T}_{u_n}$ appear in a tiling of the plane is **exactly the set of factors of the Fibonacci word** (i.e., the fixed point of the morphism $a \mapsto ab, b \mapsto a$.

Theorem (Jeandel, Rao, 2015)

The 11 Wang tile set $\mathcal{T}$ is **aperiodic**.

Some remarks/questions :

- The proof is based **on transducers**.
- It seems there is **much more to understand**.
- Can we find a **short 10 lines proof** of existence and aperiodicity ?

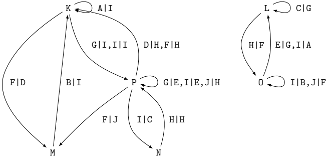
Let us make a small detour

A tile set $\mathcal{U}$ of cardinality 19

Vertical colors : $\{A, B, C, D, E, F, G, H, I, J\}$
Horizontal colors : $\{K, L, M, N, O, P\}$

$$\mathcal{U} = \left\{ \begin{array}{lllll}
u_0 = \boxed{\begin{array}{ccc} & \mathrm{L} & \\ \mathrm{C} & & \mathrm{G} \\ & \mathrm{L} & \end{array}}, &
u_1 = \boxed{\begin{array}{ccc} & \mathrm{L} & \\ \mathrm{E} & & \mathrm{G} \\ & \mathrm{O} & \end{array}}, &
u_2 = \boxed{\begin{array}{ccc} & \mathrm{O} & \\ \mathrm{H} & & \mathrm{F} \\ & \mathrm{L} & \end{array}}, &
u_3 = \boxed{\begin{array}{ccc} & \mathrm{O} & \\ \mathrm{I} & & \mathrm{B} \\ & \mathrm{O} & \end{array}}, &
u_4 = \boxed{\begin{array}{ccc} & \mathrm{L} & \\ \mathrm{I} & & \mathrm{A} \\ & \mathrm{O} & \end{array}}, \\
u_5 = \boxed{\begin{array}{ccc} & \mathrm{O} & \\ \mathrm{J} & & \mathrm{F} \\ & \mathrm{O} & \end{array}}, &
u_6 = \boxed{\begin{array}{ccc} & \mathrm{N} & \\ \mathrm{I} & & \mathrm{C} \\ & \mathrm{P} & \end{array}}, &
u_7 = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{J} & & \mathrm{H} \\ & \mathrm{P} & \end{array}}, &
u_8 = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{H} & & \mathrm{H} \\ & \mathrm{N} & \end{array}}, &
u_9 = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{I} & & \mathrm{E} \\ & \mathrm{P} & \end{array}}, \\
u_{10} = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{I} & & \mathrm{I} \\ & \mathrm{K} & \end{array}}, &
u_{11} = \boxed{\begin{array}{ccc} & \mathrm{K} & \\ \mathrm{D} & & \mathrm{H} \\ & \mathrm{P} & \end{array}}, &
u_{12} = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{G} & & \mathrm{E} \\ & \mathrm{P} & \end{array}}, &
u_{13} = \boxed{\begin{array}{ccc} & \mathrm{K} & \\ \mathrm{F} & & \mathrm{H} \\ & \mathrm{P} & \end{array}}, &
u_{14} = \boxed{\begin{array}{ccc} & \mathrm{P} & \\ \mathrm{G} & & \mathrm{I} \\ & \mathrm{K} & \end{array}}, \\
u_{15} = \boxed{\begin{array}{ccc} & \mathrm{M} & \\ \mathrm{F} & & \mathrm{J} \\ & \mathrm{P} & \end{array}}, &
u_{16} = \boxed{\begin{array}{ccc} & \mathrm{K} & \\ \mathrm{B} & & \mathrm{I} \\ & \mathrm{M} & \end{array}}, &
u_{17} = \boxed{\begin{array}{ccc} & \mathrm{K} & \\ \mathrm{A} & & \mathrm{I} \\ & \mathrm{K} & \end{array}}, &
u_{18} = \boxed{\begin{array}{ccc} & \mathrm{M} & \\ \mathrm{F} & & \mathrm{D} \\ & \mathrm{K} & \end{array}} &
\end{array} \right\}$$

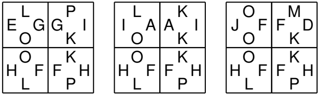
The dual (why ?) transducer representation of the tile set is below



Lemma

There are **139 valid tilings** of the 2 × 2 square by $\mathcal{U}$.

```
sage: from slabbe import WangTileSolver
sage: tiles = ['GLCL', 'GLEO', 'FOHL', 'BOIO', 'ALIO', 'FOJO', 'CNIP',
....: 'HPJP', 'HPHN', 'EPIP', 'IPIK', 'HKDP', 'EPGP', 'HKFP', 'IPGK',
....: 'JMFP', 'IKBM', 'IKAK', 'DMFK']
sage: W = WangTileSolver(map(tuple, tiles), 2, 2)
sage: W.number_of_solutions() # involves Knuth's dancing links algo
139
```



Lemma

The **set of states** of the str. con. c. of $\mathcal{U}^2$ (vert. colors) is

$$\{AF, AG, BA, BF, BG, CH, DI, EC, EH, FA, FB, \cancel{FF}, FG, GF, GG, HE, HI, IC, ID, IE, IH, II, IJ, JI\}.$$

The **set of states** of the str. con. c. of $\mathcal{U}^{*2}$ (horiz. colors) is

$$\{KO, KP, LK, LP, MK, MO, MP, NL, OK, OM, PL, PN, PO, PP\}.$$

What can go below ?

Lemma

In any tiling of $\mathbb{Z}^2$ by the tile set $\mathcal{U}$,

(i) the tile below of u_2 is u_0, u_1 or u_4,

(ii) the tile below of u_5 is u_3,

(iii) the tile below of u_7 is u_9,

(iv) the tile below of u_8 is u_6,

(v) the tile below of u_{11} is u_9 or u_{10},

(vi) the tile below of u_{13} is u_{12} or u_{14},

(vii) the tile below of u_{15} is u_{14},

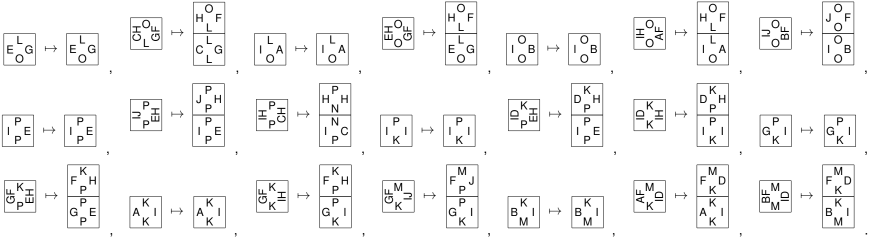
(viii) the tile below of u_{18} is u_{16} or u_{17}.

Desubstitute tiling by $\mathcal{U}$ into tilings by $\mathcal{V}$ under σ

The previous lemma allows desubstitute tilings by $\mathcal{U}$ by σ in terms of a new alphabet $\mathcal{V} = \{v_i\}_{0 \leq i \leq 20}$ of cardinality 21 :

$$\sigma = \begin{cases} v_0 \mapsto \begin{pmatrix} u_1 \end{pmatrix}, \quad v_1 \mapsto \begin{pmatrix} u_2 \\ u_0 \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} u_4 \end{pmatrix}, \quad v_3 \mapsto \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}, \quad v_4 \mapsto \begin{pmatrix} u_3 \end{pmatrix}, \\ v_5 \mapsto \begin{pmatrix} u_2 \\ u_4 \end{pmatrix}, \quad v_6 \mapsto \begin{pmatrix} u_5 \\ u_3 \end{pmatrix}, \quad v_7 \mapsto \begin{pmatrix} u_9 \end{pmatrix}, \quad v_8 \mapsto \begin{pmatrix} u_7 \\ u_9 \end{pmatrix}, \quad v_9 \mapsto \begin{pmatrix} u_8 \\ u_6 \end{pmatrix}, \\ v_{10} \mapsto \begin{pmatrix} u_{10} \end{pmatrix}, \quad v_{11} \mapsto \begin{pmatrix} u_{11} \\ u_9 \end{pmatrix}, \quad v_{12} \mapsto \begin{pmatrix} u_{11} \\ u_{10} \end{pmatrix}, \quad v_{13} \mapsto \begin{pmatrix} u_{14} \end{pmatrix}, \quad v_{14} \mapsto \begin{pmatrix} u_{13} \\ u_{12} \end{pmatrix}, \\ v_{15} \mapsto \begin{pmatrix} u_{17} \end{pmatrix}, \quad v_{16} \mapsto \begin{pmatrix} u_{13} \\ u_{14} \end{pmatrix}, \quad v_{17} \mapsto \begin{pmatrix} u_{15} \\ u_{14} \end{pmatrix}, \quad v_{18} \mapsto \begin{pmatrix} u_{16} \end{pmatrix}, \quad v_{19} \mapsto \begin{pmatrix} u_{18} \\ u_{17} \end{pmatrix}, \\ v_{20} \mapsto \begin{pmatrix} u_{18} \\ u_{16} \end{pmatrix}. \end{cases}$$

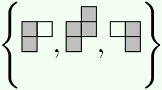
The effect of the substitution σ on tiles is



The domino and square SFT

Lemma

Let T be a tiling of $\mathbb{Z}^2$ by the vertical domino ▯ and the unit square □. The preimage of T by the substitution ■ $\mapsto$ ▯, □ $\mapsto$ □ is an **edge to edge** tiling by unit square if and only if T belongs to the **subshift of finite type** defined by the following set of forbidden patterns :



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TODO : two images (bad + good)

Desubstitute tiling by $\mathcal{U}$ into tilings by $\mathcal{V}$ under σ

Proposition

Any tiling of $\mathbb{Z}^2$ by tiles of $\mathcal{U}$ can be desubstituted by σ and the preimage is a Wang tiling of $\mathbb{Z}^2$ with the tile set $\mathcal{V}$.

Proof idea :

- We observe the left and right colors of the $\sigma(a)$:

D,F,H,J $\ni$ $\in$ D,F,H,J

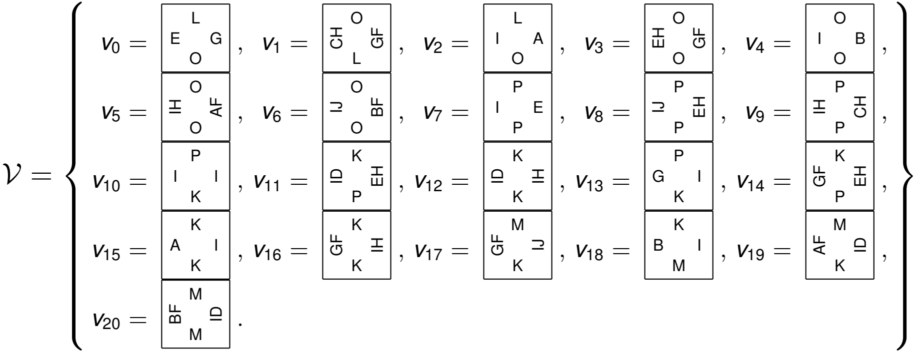
A,B,C,E,G,I $\ni$ $\in$ A,B,C,E,G,I

A,B,E,G,I $\ni$ $\in$ A,B,E,G,I

- Pattern  and pattern  are forbidden since $\{D, F, H, J\} \cap \{A, B, E, G, I\} = \varnothing$.
- Pattern  is forbidden since $\{D, F, H, J\} \cap \{A, B, C, E, G, I\} = \varnothing$.

A tile set $\mathcal{V}$ of cardinality 21

Vertical colors : $\{A, B, E, G, I, AF, BF, CH, EH, GF, ID, IH, IJ\}$
Horizontal colors : $\{K, L, M, O, P\}$.



TITLE

Lemma

In any tiling of $\mathbb{Z}^2$ by the tile set $\mathcal{V}$,

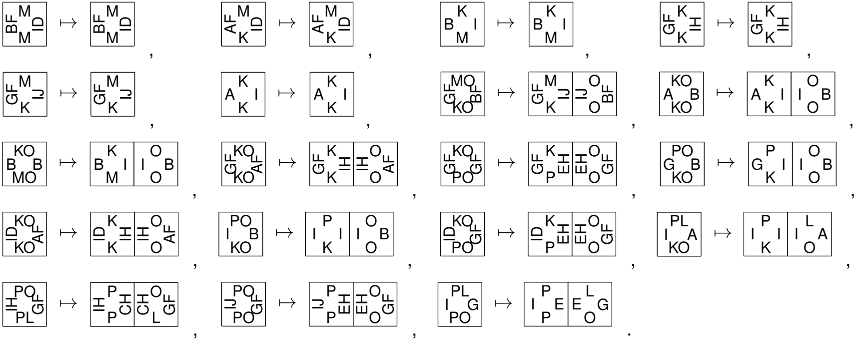
- **(i)** the tile to the left of v_0 is v_7,
- **(ii)** the tile to the left of v_1 is v_9,
- **(iii)** the tile to the left of v_2 is v_{10},
- **(iv)** the tile to the left of v_3 is v_8, v_{11} or v_{14},
- **(v)** the tile to the left of v_4 is v_{10}, v_{13}, v_{15} or v_{18},
- **(vi)** the tile to the left of v_5 is v_{12} or v_{16},
- **(vii)** the tile to the left of v_6 is v_{17}.

Desubstitute tiling by $\mathcal{V}$ into tilings by $\mathcal{W}$ under μ

The previous lemma allows desubstitute tilings by $\mathcal{V}$ under μ in terms of a new alphabet $\mathcal{W} = \{w_i\}_{0\leq i\leq 18}$ of cardinality 19 :

$$\mu = \begin{cases} w_0 \mapsto \left(v_{20} \right), & w_1 \mapsto \left(v_{19} \right), & w_2 \mapsto \left(v_{18} \right), & w_3 \mapsto \left(v_{16} \right), \\ w_4 \mapsto \left(v_{17} \right), & w_5 \mapsto \left(v_{15} \right), & w_6 \mapsto \left(v_{17} \ v_6 \right), & w_7 \mapsto \left(v_{15} \ v_4 \right), \\ w_8 \mapsto \left(v_{18} \ v_4 \right), & w_9 \mapsto \left(v_{16} \ v_5 \right), & w_{10} \mapsto \left(v_{14} \ v_3 \right), & w_{11} \mapsto \left(v_{13} \ v_4 \right), \\ w_{12} \mapsto \left(v_{12} \ v_5 \right), & w_{13} \mapsto \left(v_{10} \ v_4 \right), & w_{14} \mapsto \left(v_{11} \ v_3 \right), & w_{15} \mapsto \left(v_{10} \ v_2 \right), \\ w_{16} \mapsto \left(v_9 \ v_1 \right), & w_{17} \mapsto \left(v_8 \ v_3 \right), & w_{18} \mapsto \left(v_7 \ v_0 \right). & \end{cases}$$

The effect of the substitution μ on tiles is



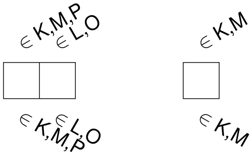
Desubstitute tiling by $\mathcal{V}$ into tilings by $\mathcal{W}$ under μ

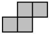
Proposition

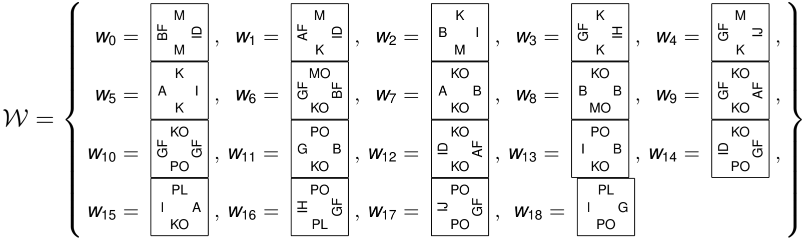
Any tiling of $\mathbb{Z}^2$ by tiles of $\mathcal{V}$ can be desubstituted by μ and the preimage is a Wang tiling of $\mathbb{Z}^2$ with the tile set $\mathcal{W}$.

Proof idea :

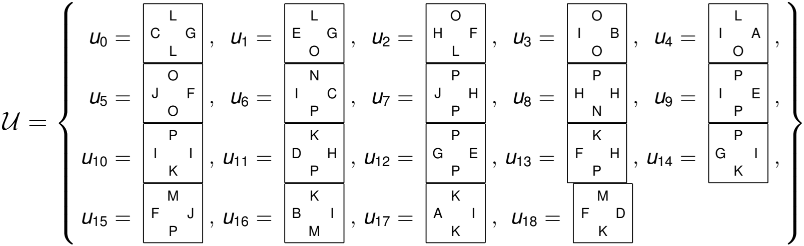
- We observe the bottom and top colors of the $\mu(a)$:



- Pattern  and pattern  are forbidden since $\{L, O\} \cap \{K, M\} = \varnothing$.
- Pattern  is forbidden since $\{L, O\} \cap \{K, M, P\} = \varnothing$.



But recall :



Theorem

Any tiling of $\mathbb{Z}^2$ by the tile set $\mathcal{U}$ is **self-similar** and **aperiodic**.

The substitution

In terms of the $\{u_i\}_{0\leq i\leq 18}$, the substitution $\sigma \circ \mu$ is

$$\begin{array}{llll}
u_0 \mapsto \begin{pmatrix} u_{18} \\ u_{16} \end{pmatrix}, & u_1 \mapsto \begin{pmatrix} u_{18} \\ u_{17} \end{pmatrix}, & u_2 \mapsto \begin{pmatrix} u_{16} \end{pmatrix}, & u_3 \mapsto \begin{pmatrix} u_{13} \\ u_{14} \end{pmatrix}, \\
u_4 \mapsto \begin{pmatrix} u_{15} \\ u_{14} \end{pmatrix}, & u_5 \mapsto \begin{pmatrix} u_{17} \end{pmatrix}, & u_6 \mapsto \begin{pmatrix} u_{15} & u_5 \\ u_{14} & u_3 \end{pmatrix}, & u_7 \mapsto \begin{pmatrix} u_{17} & u_3 \end{pmatrix}, \\
u_8 \mapsto \begin{pmatrix} u_{16} & u_3 \end{pmatrix}, & u_9 \mapsto \begin{pmatrix} u_{13} & u_2 \\ u_{14} & u_4 \end{pmatrix}, & u_{10} \mapsto \begin{pmatrix} u_{13} & u_2 \\ u_{12} & u_1 \end{pmatrix}, & u_{11} \mapsto \begin{pmatrix} u_{14} & u_3 \end{pmatrix}, \\
u_{12} \mapsto \begin{pmatrix} u_{11} & u_2 \\ u_{10} & u_4 \end{pmatrix}, & u_{13} \mapsto \begin{pmatrix} u_{10} & u_3 \end{pmatrix}, & u_{14} \mapsto \begin{pmatrix} u_{11} & u_2 \\ u_9 & u_1 \end{pmatrix}, & u_{15} \mapsto \begin{pmatrix} u_{10} & u_4 \end{pmatrix}, \\
u_{16} \mapsto \begin{pmatrix} u_8 & u_2 \\ u_6 & u_0 \end{pmatrix}, & u_{17} \mapsto \begin{pmatrix} u_7 & u_2 \\ u_9 & u_1 \end{pmatrix}, & u_{18} \mapsto \begin{pmatrix} u_9 & u_1 \end{pmatrix}. &
\end{array}$$

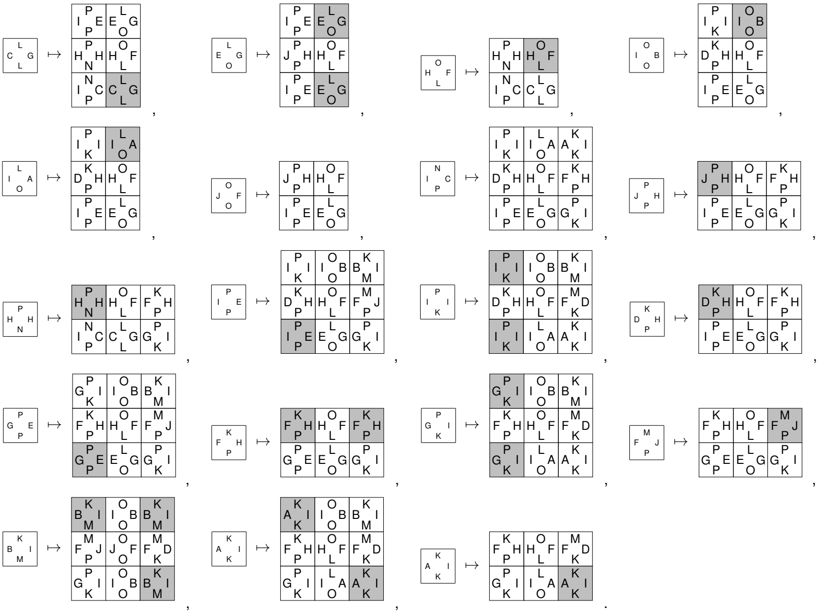
Proposition

$\sigma \circ \mu$ is primitive. Its characteristic polynomial is

$$\chi_M(x) = x^3 \cdot (x-1)^4 \cdot (x+1)^4 \cdot (x^2 - 3x + 1) \cdot (x^2 + x - 1)^3.$$

The PF eigenvalue is $\varphi^2 = \varphi + 1 = (3+\sqrt{5})/2$.

$(\sigma \circ \mu)^2$ is prolongable



Back to the main road...

Wang tiles from codings of $\mathbb{Z}^2$-actions

- Let D be a **set**,
- $u, v : D \to D$ two **invertible transformations** s.t. $u \circ v = v \circ u$,
- I and J : two finite not necessarily disjoint **sets of colors**,
- $D = \cup_{i \in I} X_i$ and $D = \cup_{j \in J} Y_j$ be two **partitions** of D.

This gives the **left and bottom colors** :

$$\begin{array}{rccl} \ell : & D & \to & I \\ & \mathbf{x} & \mapsto & i \text{ if } \mathbf{x} \in X_i, \end{array} \qquad \begin{array}{rccl} b : & D & \to & J \\ & \mathbf{x} & \mapsto & j \text{ if } \mathbf{x} \in Y_j. \end{array}$$

and the **right and top colors** $r : D \to I$, $t : D \to J$ as :

$$r = \ell \circ u \quad \text{and} \quad t = b \circ v,$$

that is, the right color of an element $\mathbf{x} \in D$ is the left color of $u(\mathbf{x})$.
The **Wang tile coding** :

$$\begin{array}{rccl} c : & D & \to & I \times J \times I \times J \\ & \mathbf{x} & \mapsto & (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x})). \end{array}$$

Let $\mathcal{T} = c(D)$ be the associated **Wang tile set**.

Wang tilings from codings of $\mathbb{Z}^2$-actions

Lemma

Let

$$\begin{array}{rccl} f: & D & \to & \mathcal{T}^{\mathbb{Z}^2} \\ & \mathbf{x} & \mapsto & f_{\mathbf{x}} : (m,n) \mapsto c(u^m v^n \mathbf{x}). \end{array}$$

For every $\mathbf{x} \in D$, $f_{\mathbf{x}}$ is a **Wang tiling of the plane**.
Moreover the $\mathbb{Z}^2$-action of u and v on D is conjugate through f to $\mathbb{Z}^2$-translations of the tilings.

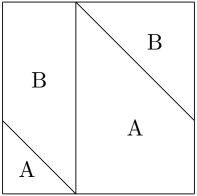
Therefore, unique ergodicity of the $\mathbb{Z}^2$-action on D will imply uniform patch frequencies in the tilings generated by f.

Codings of $\mathbb{Z}^2$-actions : Boring Example 1

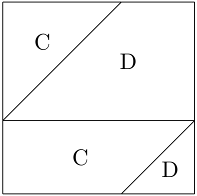
Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the **torus** $\mathbb{R}^2/\mathbb{Z}^2$, we consider the **translations**

$$u(\mathbf{x}) = \mathbf{x} + (\varphi, 0) \quad \text{and} \quad v(\mathbf{x}) = \mathbf{x} + (0, \varphi)$$

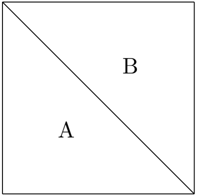
right $r(\mathbf{x})$



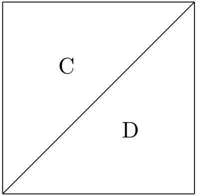
top $t(\mathbf{x})$



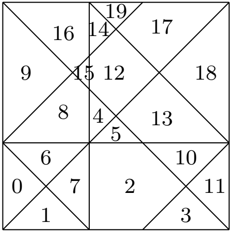
left $\ell(\mathbf{x})$



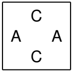
bottom $b(\mathbf{x})$

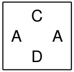
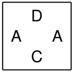


$c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$



For every $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$, $f_{\mathbf{x}} : \mathbb{Z}^2 \to \mathcal{S}$ is a **Wang tiling of the plane**.

Boring because $c(\mathbb{R}^2/\mathbb{Z}^2)$ admits periodic tilings : $u_0 =$ ,

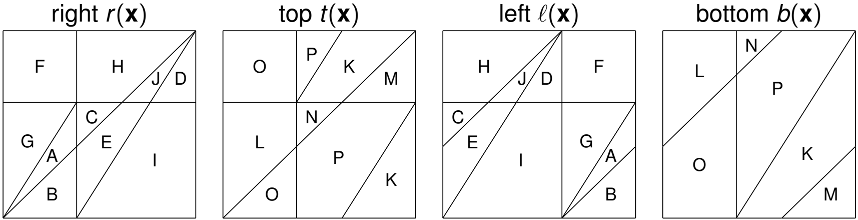
$u_1 =$ , $u_4 =$ , etc.

Codings of $\mathbb{Z}^2$-actions : Example 2

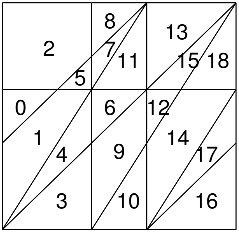
Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the **torus** $\mathbb{R}^2/\mathbb{Z}^2$, we consider the **translations**

$$u(\mathbf{x}) = \mathbf{x} + (\varphi, 0) \quad \text{and} \quad v(\mathbf{x}) = \mathbf{x} + (0, \varphi)$$

and the codings :


right $r(\mathbf{x})$ top $t(\mathbf{x})$ left $\ell(\mathbf{x})$ bottom $b(\mathbf{x})$

$$c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$$



Theorem

We have $c(\mathbb{R}^2/\mathbb{Z}^2) = \mathcal{U}$.

$u_0 = \boxed{\begin{array}{ccc} & L & \\ C & & G \\ & L & \end{array}}$, $u_1 = \boxed{\begin{array}{ccc} & L & \\ E & & G \\ & O & \end{array}}$, $u_2 = \boxed{\begin{array}{ccc} & O & \\ H & & F \\ & L & \end{array}}$,

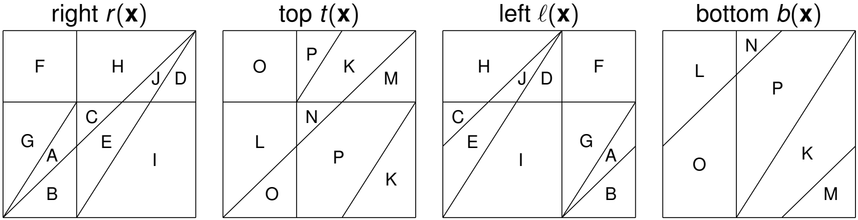
etc.

Codings of $\mathbb{Z}^2$-actions : Example 2

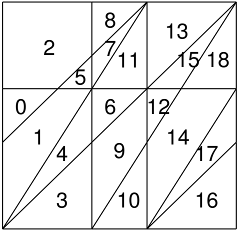
Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the **torus** $\mathbb{R}^2/\mathbb{Z}^2$, we consider the **translations**

$$u(\mathbf{x}) = \mathbf{x} + (\varphi, 0) \quad \text{and} \quad v(\mathbf{x}) = \mathbf{x} + (0, \varphi)$$

and the codings :



$$c(\mathbf{x}) = (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x}))$$



Theorem

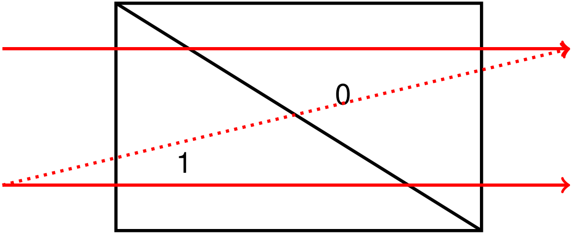
For every $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$, $f_{\mathbf{x}} : \mathbb{Z}^2 \to \mathcal{U}$ is a **Wang tiling of the plane**.

In last slide of Michaël Rao's talk, Spring, 2017

Michaël wrote an observation that **is not** in their preprint :

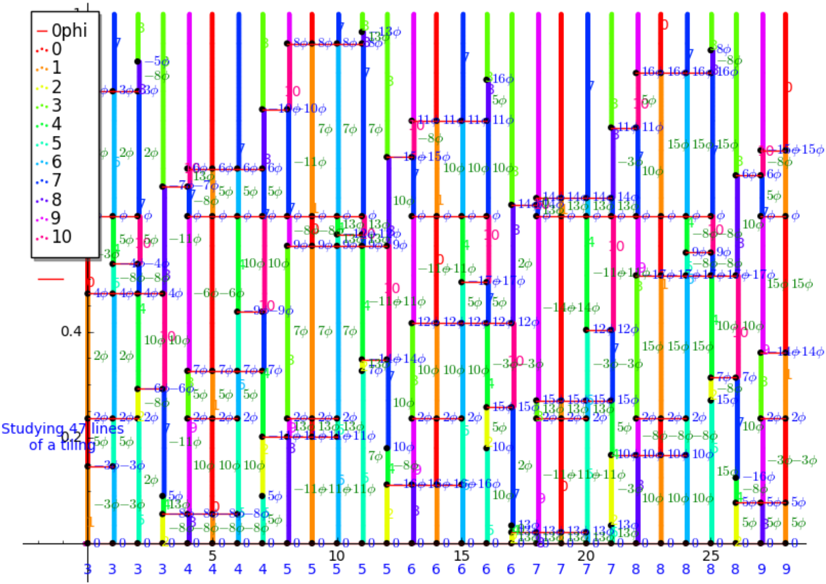
> *"Open question 2 : "proof from the book" ? If we look at densities of 1 on each line on an infinite tiling, one tranducer* ***add*** $\varphi - 1$ *and the other* ***add*** $\varphi - 2$*."*

His remark made me think about **codings of rotations** :



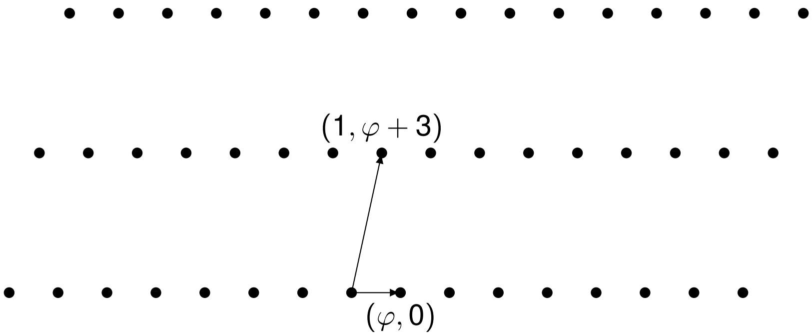
Michaël accepted to give me a 2583×986 **patch** of $a's$ and $b's$ for which I am very thankful.
... and I never look back.

Wrapping lines of Jeandel Rao patch on circles



Codings of $\mathbb{Z}^2$-actions : Example 3

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Consider the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$.



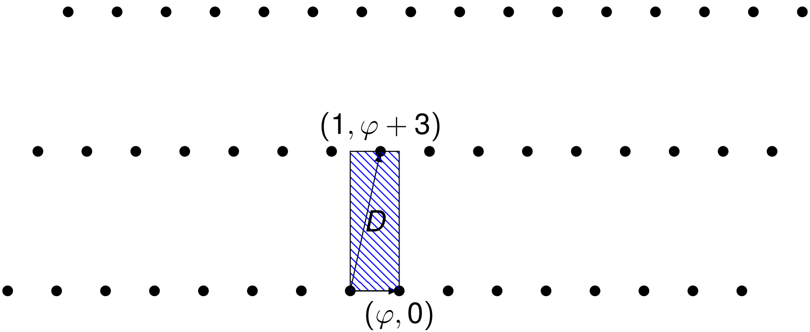
On the **torus** $\mathbb{R}^2/\Gamma$, we consider the **translations**

$$u : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x + 1, y) \end{array} \quad \text{and} \quad v : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x, y + 1). \end{array}$$

Codings of $\mathbb{Z}^2$-actions : Example 3

A **fundamental domain** of $\mathbb{R}^2/\Gamma$ is

$$D = [0, \varphi[\times [0, \varphi + 3[.$$

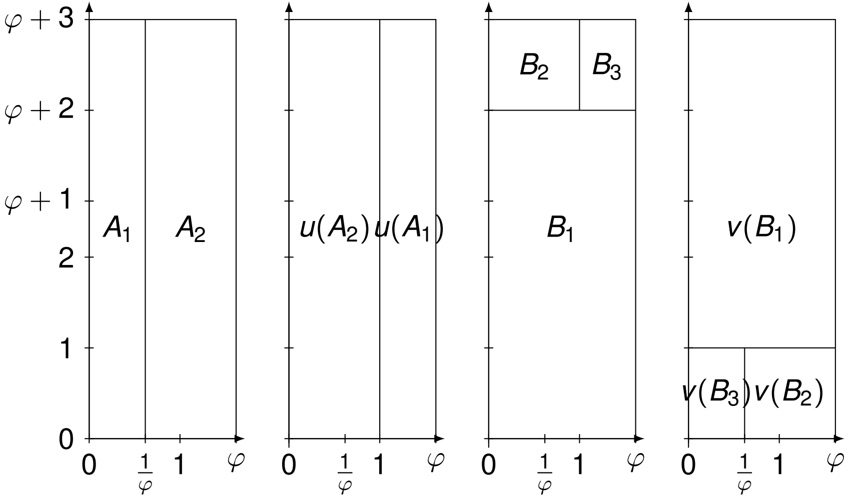


After renormalisation of transformations *u* and *v* on the torus $\mathbb{R}^2/\mathbb{Z}^2$, we observe that each translation vect. is not rationally independent :

$$\begin{pmatrix} \phi & 1 \\ 0 & \phi + 3 \end{pmatrix}^{-1} = \begin{pmatrix} \phi - 1 & -\frac{4}{11}\phi + \frac{5}{11} \\ 0 & -\frac{1}{11}\phi + \frac{4}{11} \end{pmatrix}.$$

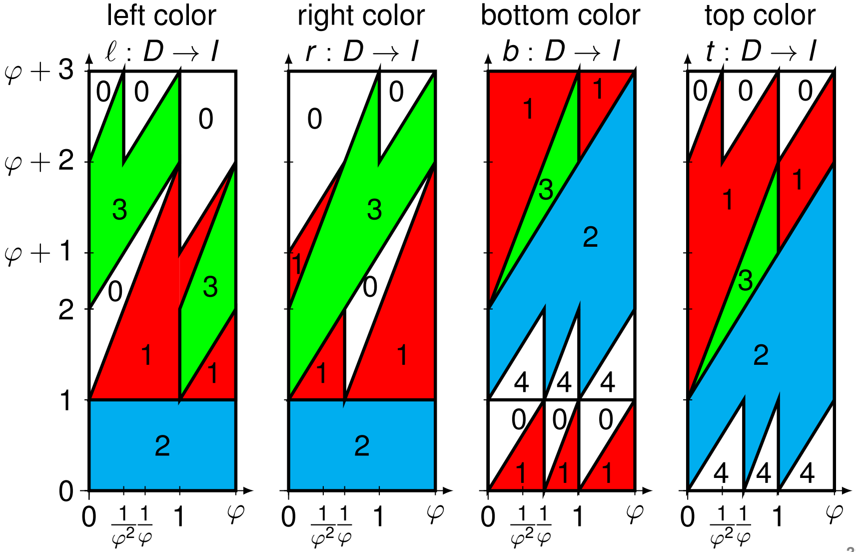
Codings of $\mathbb{Z}^2$-actions : Example 3

Transformations *u* and *v* are one-to-one **piecewise translations** of pieces on the fundamental domain *D*.



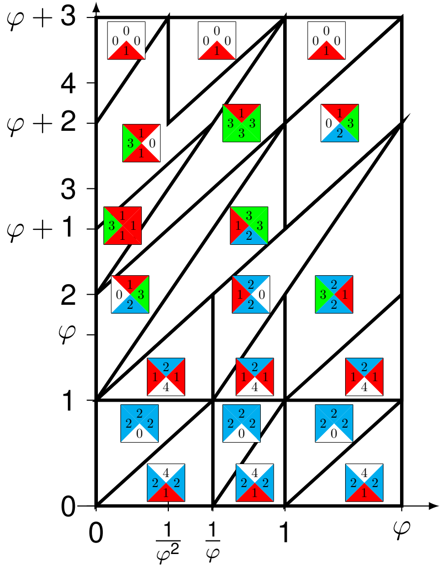
Codings of $\mathbb{Z}^2$-actions : Example 3

The **left, right bottom and top color codings** satisfying $r = \ell \circ u$ and $t = b \circ v$.



Codings of $\mathbb{Z}^2$-actions : Example 3

We deduce the **tile coding** $c : D \to \mathcal{T}$.



Theorem

We have

$$c(\mathbb{R}^2/\Gamma) = \mathcal{T}$$

where $\mathcal{T}$ is the **Jeandel-Rao tile set**.

Theorem

For every $\mathbf{x} \in \mathbb{R}^2/\Gamma$,

$$f_{\mathbf{x}} : \mathbb{Z}^2 \to \mathcal{T}$$

is a **Jeandel-Rao Wang tiling of the plane**.

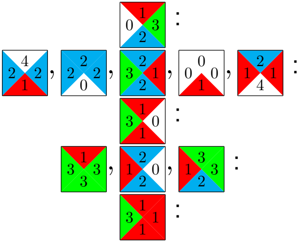
Frequency of patterns

Corollary

Since **Lebesgue** measure is the **only invariant** mesure on $\mathbb{R}^2/\Gamma$ which is invariant under both translations u and v, we have **unique ergodicity** of the tiling space

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}}$$

from which we deduce existence of pattern frequencies.



$5/(12\varphi + 14) \approx 0.1496$

$1/(2\varphi + 6) \approx 0.1083$

$1/(5\varphi + 4) \approx 0.0827$

$1/(8\varphi + 2) \approx 0.0669$

$1/(18\varphi + 10) \approx 0.0256$

Are these codings a complete description ?

Question

If Ω is the **Wang subshift** of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$, do we have

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}} = \Omega \quad ?$$

Are these codings a complete description ?

Question

If Ω is the **Wang subshift** of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$, do we have

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}} = \Omega \quad ?$$

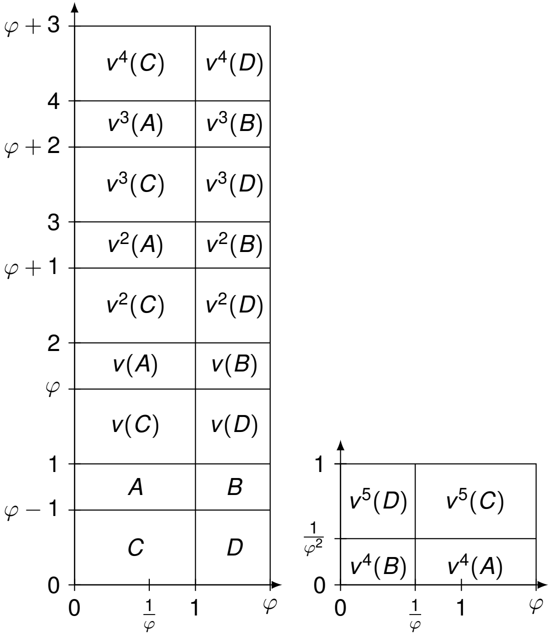
Short answer :

- In September 2017, I believe yes...
- In October 2017, I believe no...
- In November 2017, I believe yes...
- In December 2017, I will finish the proof that yes.

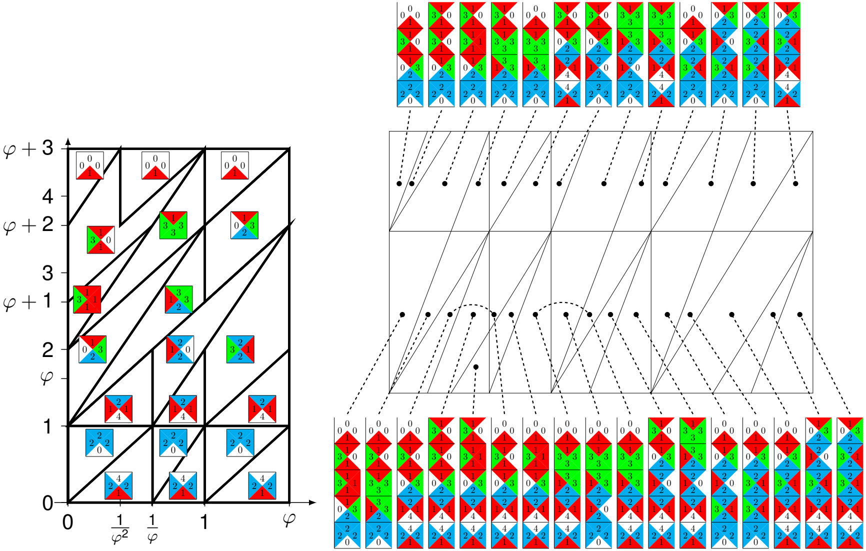
Question

Same question for $\mathcal{U}$.

September : Induction of v on $[0, \varphi[\times[0, 1[$

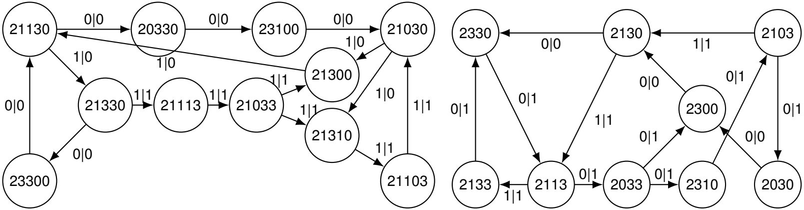


October : Induced coding on $[0,\varphi[\times[0,1[$

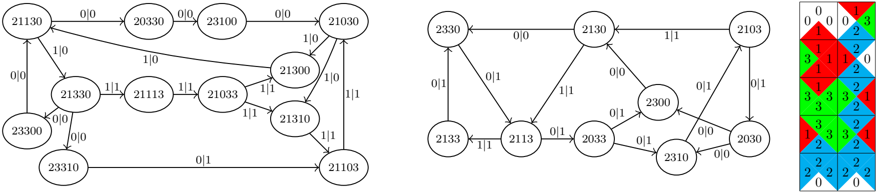


which yield 13 of 1×4 sticks $+15$ of 1×5 sticks $=28$ sticks $\neq 31$? ? ?

The transducer that we get from the induction of v on $[0,\varphi[\times[0,1[$ is :



Recall Figure 7 (b) page 17 of Jeandel-Rao preprint :

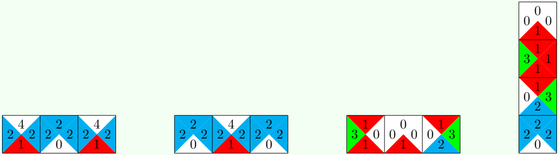


In $\overline{f(D)}$, patterns 21330 $\xrightarrow{0|0}$ 23310 $\xrightarrow{0|1}$ 21103 and 2030 $\xrightarrow{0|0}$ 2310 have frequency zero... (why ?).

A surrounding of 2030 $\xrightarrow{0|0}$ 2310

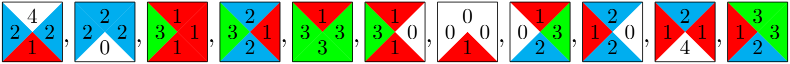
Lemma

No tiling of the plane with Jeandel-Rao tile set contains :



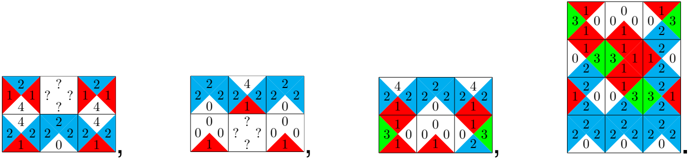
Recall

$\mathcal{T} = \{$



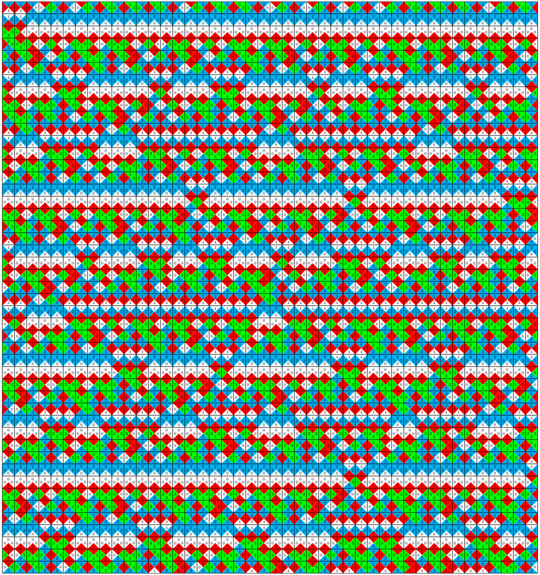
$\}$.

Proof :



A surrounding of $21330 \xrightarrow{0|0} 23310 \xrightarrow{0|1} 21103$

of radius 21 :



A surrounding of 21330 $\xrightarrow{0|0}$ 23310 $\xrightarrow{0|1}$ 21103

During a visit of Rao at LaBRI, Bordeaux :

Proposition (Michaël Rao, Nov. 2017)

No tiling of the plane with Jeandel-Rao tile set contains the pattern 21330 $\xrightarrow{0|0}$ 23310 $\xrightarrow{0|1}$ 21103.

Rao's proof :

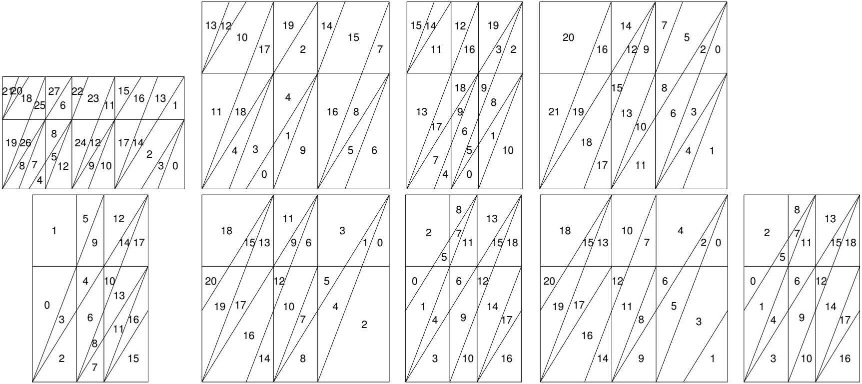
Image credit : Le Bagger 288,

`http://i.imgur.com/YH9xX.jpg`

Another proof :

```
sage: from slabbe import WangTileSolver
sage: tiles = [(2300, 0, 2030, 0), (2300, 1, 2033, 0), (2310, 1, 2033, 0), (2030, 1, 2103, 0), (2130, 1, 2103, 1),
....: (2033, 1, 2113, 0), (2133, 1, 2113, 1), (2113, 1, 2130, 1), (2330, 0, 2130, 0), (2330, 1, 2133, 0),
....: (2130, 0, 2300, 0), (2103, 1, 2310, 0), (2113, 1, 2330, 0), (23100, 0, 20330, 0), (21300, 0, 21030, 1),
....: (21310, 0, 21030, 1), (21300, 1, 21033, 1), (21310, 1, 21033, 1), (21030, 1, 21103, 1), (21033, 1, 21113, 1),
....: (20330, 0, 21130, 0), (21330, 0, 21130, 1), (21130, 0, 21300, 1), (21103, 1, 21310, 1), (21113, 1, 21330, 1),
....: (23300, 0, 21330, 0), (23310, 0, 21330, 0), (21030, 0, 23100, 0), (21130, 0, 23300, 0), (21103, 1, 23310, 0)]
sage: center = (35,4)
sage: right  = {center:21103}; left = {center:23310}; top = {center:1}; bottom = {center:0}
sage: W = WangTileSolver(tiles, width=71, height=9, preassigned=[right,top,left,bottom])
sage: p,x = W.milp(); p
Boolean Program (maximization, 19170 variables, 1841 constraints)
sage: tiling = W.solve(solver='Gurobi')           # < 20 min. on a normal 2017 desktop computer with 8 cpus
...
MIPSolverException: Gurobi: The problem is infeasible
```

On going work : A series of inductions



Theorem

The coding of the $\mathbb{Z}^2$-action of u and v on $D = \mathbb{R}^2/\Gamma$ (Jeandel Rao) is **conjugate** to the coding of the $\mathbb{Z}^2$-action of u and v on $\mathbb{R}^2/\mathbb{Z}^2$ (the self-similar one).

On going work : A series of inductions

Conjecture (or Theorem to be written soon)

The Jeandel-Rao tilings dynamical system by $\mathcal{T}$ can be **desubstituted uniquely** into the self-similar tilings dynamical system by $\mathcal{U}$ (a sequence of 7 substitutions).

Corollary (to be deduced soon)

If Ω is the **Wang subshift** of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$, then

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}} = \Omega.$$

The Open Questions

- Find the **10 line proof** for the aperiodicity of Jeandel-Rao tilings.
- How does it **compare** to other aperiodic tilings ?
- Can we generalize Jeandel-Rao tilings to **other Pisot numbers** ?
- I think the fundamental domain can be identified with the **window** of a cut-and-project set with dimension $2+2$.
- What are the structure of the **other aperiodic tile sets** of cardinality 11 found by Jeandel-Rao ?
- Does there **exists an aperiodic self-similar** Wang tile set of cardinality less than 19 ?
- Understand all of this in terms of **shape-symmetric** multidim. sequences (Maes, 1999) and S-automatic sequences using

Theorem (Charlier, Kärki, Rigo, 2010)

Let $d \geq 1$. The d-dimensional infinite word x is **S-automatic** for some abstract numeration system $S = (L, \Sigma, <)$ where $\epsilon \in L$ if and only if x is the **image by a coding of a shape-symmetric** d-dimensional infinite word.

Sharing my Sage code

I wrote **three Python modules** while working on this project :

```
sage: from slabbe import WangTileSet, WangTileSolver
sage: from slabbe import Substitution2d
sage: from slabbe import PolyhedronPartition
```

The code is **open-source** and is available in the develop branch of

`https://github.com/seblabbe/slabbe`

It will be **part of the next release** of the optional Sage package `slabbe-0.3`. Currently, one can **install** the previous version (0.2) :

```
sage -pip install slabbe
```

Lastly, **many thanks** to the new LaTeXcommand I just learned about :

```
\resizebox{\linewidth}{!}{
    % overfull horizontal lines of pictures or equations
}
```